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CERTAIN PROPERTIES OF SOFT MULTI-SET TOPOLOGY WITH APPLICATIONS IN MULTI-CRITERIA DECISION MAKING

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Abstract *Original scientific paper*

The aim of this paper is to introduce the notion of soft multi-set topology (SMS-topology) defined on a soft multi-set (SMS). Soft multi-set and soft multi-set topology are fundamental tools in computational intelligence, which have a large number of applications in soft computing, fuzzy modeling and decision-making under uncertainty. The idea of power whole multi-subsets of a SMS is defined to explore various rudimentary properties of SMS-topology. Certain properties of SMS-topology like SMS-basis, SMS-subspace, SMS-interior, SMS-closure and boundary of SMS are explored. Furthermore, the multi-criteria decision-making (MCDM) algorithms with aggregation operators based on SMS-topology are established. Algorithm i $(i = 1, 2, 3)$ are developed for the selection of best alternative for biopesticides, for the selection of best textile company, for the award of performance, respectively. Some real life applications of the proposed algorithms in MCDM problems are illustrated by numerical examples. The the reliability and feasibility of proposed MCDM techniques is shown by comparison analysis with some existing techniques.

Keywords: Soft multi-sets; soft multi-set topology; aggregation operators, algorithms; MCDM.

1 Introduction

Modeling and handling uncertainties has become an issue of great importance in the solution of sophisticated problems originating in a vast range of various fields such as computational intelligence, artificial intelligence, data analysis, information fusion, image processing, signal processing, environmental sciences and medical sciences. Mathematical models like multi-sets (Blizard, 1989), fuzzy sets (Zadeh, 1965), soft sets (Molodtsov, 1999) and rough sets (Pawlak, 1982) are fundamental tools for uncertainty, hesitancy and vagueness in the real life circumstances. The researchers have been developed some extension of fuzzy sets like intuitionistic fuzzy sets (Atanassov, 1986), bipolar fuzzy sets (Zhang, 1994), Pythagorean fuzzy sets (Yager, 2013; Yager and Abbasov, 2013) and q-rung orthopair fuzzy sets (Yager, 2017) which have a large number of applications

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in computational intelligence, decision making under uncertainty and many other fields of science and engineering. Indeed, the real power of these sets are in their ability to handle and manipulate verbally-stated information into mathematical modeling and seeking feasible solutions to complicated real life problems. Additionally, fuzzy sets and its extensions are strong mathematical models to solve real world problems which can not be solved by classical mathematical techniques.

Fuzzy sets, extensions of fuzzy sets, rough sets, soft sets and hybrid structures of these sets have been studied by many researchers like Ali, (2009,2011); Cagman *et al.*, (2011); Chen (2005); Feng *et al.*, (2010,2011,2018); Garg and Rani, (2019); Hashmi *et al.*, (2019); Karaaslan and Hunu, (2020); Kumar and Garg, (2018), Maji *et al.*, (2002,2003); Naeem *et al.*, (2019), Peng and Yang (2015), Peng *et al.*, (2017), Pie and Miao (2005), Roy and Maji (2007); Riaz *et al.*, (2019);, Riaz and Hashmi (2019); Riaz and Tehrim, (2019); Shabir and Naz (2011); Zhang and Xu (2014); Zhan *et al.*, (2015,2019); and Zhang (1994).

Multi-set theory and soft multi-set theory have been studied by many researchers including Alkhazaleh *et al.* (2011); Babitha and John (2013); Balami and Ibrahim (2013); Girish and John (2009,2019); Kumar and Naisal (2016); Mukherjee *et al.* (2014); Syropoulos (2001) and Tokat and Osmanoglu (2011,2013).

A large number of MCDM methods have been developed by the researchers under rough sets, fuzzy sets and soft sets. But these methods do not deal with real life situations under the universe of soft multi-sets. Due to repetition of objects or objects have multiplicity more than one and variety of attributes under consideration in the universe of soft multi-sets it is necessary to develop novel MCDM approaches. The goal of this article is deal with these challenges and to extend the notion of soft multi-sets and soft multi-set topology towards MCDM problems. The topological and algebraic structures of soft multi-sets have large number of applications in soft computing, decision-making, data analysis, data mining, expert systems, information aggregation and information measures.

The remaining article is arranged as follows: In section 2, we use power whole multi-subsets of a SMS to introduce some basic concepts of SMS-theory. In section 3, we present some new results of SMS-topology and certain properties including basis, subspace, interior, closure and boundary of soft multi-sets (SMSs). In Section 4, we present Algorithm 1, Algorithm 2 and Algorithm 3 for the selection of best alternative for biopesticides, for the selection of best textile company, for the award of performance, respectively. We also present applications of SMS-topology for MCDM by using proposed algorithms. At the end, the sum up of this research studies is given in the in Section 5.

2 Preliminaries

In this section, we study few primary rudiments of multi-sets (MSs) and soft multi-sets (SMSs).

Definition 2.1. "A multi-set (MS) over Z is just a pair $< Z, f >$, where $f : Z \to W$ is a function, Z is a crisp set and W is a set of whole numbers. Also in order to avoid any confusion we will use square brackets for multi-sets and braces for sets. Multiset A is given by $A =< Z, f >= [\frac{k_1}{z_1}, \frac{k_2}{z_2}, ..., \frac{k_n}{z_n}]$, where z_1 occuring k_1 times, z_2 occuring k_2 times and so on (Syropoulos, 2001).

Definition 2.2. Let $A =< Z, f >$ and $B =< Z, g >$ be two multi-sets. Multiset A is a submulti-set of B, denoted by $A \subseteq B$ if for all $z \in A$, $f(z) \leq g(z)$ (Syropoulos, 2001).

Definition 2.3. A submulti-set $A =< Z, f >$ of $B =< Z, g >$ is a whole submulti-set of B with each element in A having full multiplicity as in B. i.e. $f(z) = g(z)$, for every z in A (Babitha and John (2013))

Definition 2.4. Let $[Z]^n$ denotes the set of all MSs whose elements are in Z such that no element in a multi-set appears more than n times. Let $A \in [Z]^n$ be a multi-set. The power whole multi-set of A denoted by $PW(A)$ is defined as the set of all whole sub MSs of A. The cardinality of $PW(A)$ is 2^m, where m is the cardinality of the support set (root set) of A" (Babitha and John (2013)).

In the sequel, H indicates to universal multi-set, E is a set of attributes or parameters , $PW(H)$ is a power whole multi-set of H and $A \subseteq E$.

Example 2.5. Let $\dddot{M} = [2/r, 1/y, 1/k]$ be a multi-set. Then the set of all sub MSs of M is

$$PW(A) = \Big\{ \dddot{M}_1 = [0/r, 0/y, 0/k],\ \dddot{M}_2 = [0/r, 0/y, 1/k],\ \dddot{M}_3 = [0/r, 1/y, 0/k],\ \dddot{M}_4 = [0/r, 1/y, 1/k],$$

$$\dddot{M}_5 = [1/r, 0/y, 0/k],\ \dddot{M}_6 = [1/r, 0/y, 1/k],\ \dddot{M}_7 = [1/r, 1/y, 0/k],\ \dddot{M}_8 = [1/r, 1/y, 1/k],$$

$$\dddot{M}_9 = [2/r, 0/y, 0/k],\ \dddot{M}_{10} = [2/r, 0/y, 1/k],\ \dddot{M}_{11} = [2/r, 1/y, 0/k],\ \dddot{M}_{12} = [2/r, 1/y, 1/k] \Big\}$$

and $card(P(M)) = (2 + 1)(1 + 1)(1 + 1) = 12$.

Furthermore, the power whole multi-set is given by

$$PW(M) = \{\dddot{M}_1,\ \dddot{M}_2,\ \dddot{M}_3,\ \dddot{M}_4,\ \dddot{M}_9,\ \dddot{M}_{10},\ \dddot{M}_{11},\ \dddot{M}_{12}\}$$

and its cardinality is given by $card(PW(M)) = 2^3 = 8$.

Definition 2.6. "A soft multi-set (SMS) Ω_A on the universal multi-set H is defined by the set of all ordered pairs $\Omega_A = \{(\nu, \Omega_A(\nu)) : \nu \in E, \Omega_A(\nu) \in PW(H)\}$, where $\Omega_A : E \to PW(H)$ such that $\Omega_A(\nu) = \emptyset$ if $\nu \notin A$.

Throughout this paper, $SM(H)$ denotes the family of all SMSs over H with attributes from E. Now, we elaborate the definition of soft multi-set by the succeeding example" (Babitha and John (2013)).

Example 2.7. Let $H = [\frac{2}{r_1}, \frac{4}{r_2}, \frac{3}{r_3}, \frac{5}{r_4}, \frac{7}{r_5}, \frac{6}{r_6}, \frac{9}{r_7}]$ be the universal multi-set of classrooms,

$$E = \{\text{comfortable, air conditioned, well decorated, flipped classroom}\}$$

and $A = E$. Then the SMS Ω_A is given by

$$\Omega_A = \{(\text{comfortable}, [\tfrac{2}{r_1}, \tfrac{5}{r_4}]), (\text{air conditioned}, [\tfrac{6}{r_6}, \tfrac{9}{r_7}]),$$
$$(\text{well decorated}, [\tfrac{2}{r_1}, \tfrac{4}{r_2}]), (\text{flipped classroom}, [\tfrac{3}{r_3}, \tfrac{7}{r_5}, \tfrac{9}{r_7}])\}.$$

Definition 2.8. "Let $\Omega_A \in SM(H)$. If $\Omega_A(\nu) = \emptyset$, $\forall\ \nu \in E$, then Ω_A is called an empty or null SMS, denoted by Ω_ϕ (See Babitha and John (2013)).

Definition 2.9. Let $\Omega_A \in SM(H)$. Then Ω_A is said to be A-universal SMS, denoted by $\Omega_{\widehat{A}}$, if $\Omega_A(\nu) = H$, $\forall\ \nu \in A$. If $A = E$, then A-universal soft multi-set is said to be an universal or absolute SMS, denoted by $\Omega_{\widehat{E}}$" (Babitha and John (2013)).

Definition 2.10. Let $\Omega_A,\ \Omega_B \in SM(H)$. Then, Ω_A is a soft multi subset of Ω_B, denoted by $\Omega_A \widehat{\subseteq} \Omega_B$, if $\Omega_A(\nu) \subseteq \Omega_B(\nu)$ for all $\nu \in E$" (Babitha and John (2013)).

Definition 2.11. Let $\Omega_A,\ \Omega_B \in EM(H)$. Then, the union $\Omega_A \widehat{\cup} \Omega_B$, the intersection $\Omega_A \widehat{\cap} \Omega_B$, the difference $\Omega_A \widehat{\setminus} \Omega_B$ of Ω_A and Ω_B are defined by the approximate functions $\Omega_{A\widehat{\cup}B}(\nu) = \Omega_A(\nu) \cup \Omega_B(\nu)$, $\Omega_{A\widehat{\cap}B}(\nu) = \Omega_A(\nu) \cap \Omega_B(\nu)$, $\Omega_{A\widehat{\setminus}B}(\nu) = \Omega_A(\nu) \ominus \Omega_B(\nu)$, respectively, and the complement Ω_A^c of Ω_A is defined $\Omega_A^c(\nu) = H \ominus \Omega_A(\nu)$, for all $\nu \in E$. Note that $(\Omega_A^c)^c = \Omega_A$ and $\Omega_\phi^c = \Omega_{\widehat{E}}$.

Definition 2.12. A soft multi-set Ω_A over H is called soft multi-set point (SMS-point), if there is exactly one $\nu \in A$, such that $\Omega_A(\nu) \neq \emptyset$ and $\Omega_A(\mu) = \emptyset$, $\forall \mu \in A \setminus \{\nu\}$. The SMS-point Ω_A is in the SMS δ_A, if for the element $\nu \in A,\ \Omega_A(\nu) \subseteq \delta_A(\nu)$.

Example 2.13. Let $H = [\frac{2}{a}, \frac{3}{b}, \frac{4}{c}]$, $A = \{\nu, \mu\} = E$. Let $\Omega_A = \{(\nu, [\frac{2}{a}])\}$ and $\delta_A = \{(\nu, [\frac{2}{a}, \frac{3}{b}]), (\mu, [\frac{3}{b}, \frac{4}{c}])\}$.
Since $\Omega_A(\nu) = [\frac{2}{a}] \subseteq [\frac{2}{a}, \frac{3}{b}] = \delta_A(\nu)$ and $\Omega_A(\mu) = \emptyset\ \forall \mu \in A \setminus \{\nu\}$.
Therefore, Ω_A is a SMS-point of SMS δ_A. , where

Proposition 2.14. *Let* $\Omega_A, \Omega_B \in SM(H)$. *Then*
(i) $(\Omega_A \widehat{\cup} \Omega_B)^c = \sigma_A^c \widehat{\cap} \sigma_B^c$,
(ii) $(\Omega_A \widehat{\cap} \Omega_B)^c = \sigma_A^c \widehat{\cup} \sigma_B^c$.

3 Soft Multi-Set Topology

Different approaches have bee studied by the researchers to define soft multi-set topology (SMS-topology) (Mukherjee *et al.* (2014), and Tokat and Osmanoglu (2011,2013)). In this section, we introduce the notion of SMS-topology on a soft multi-set and its analogous properties by using the concept of power whole sub multi-sets to use the full multiplicity or zero multiplicity of each objects.

Definition 3.1. Let Ω_A be a SMS over H. The soft power whole multi-set of the SMS Ω_A is denoted by $\widetilde{PW}(\Omega_A)$ and is defined as

$$\widetilde{PW}(\Omega_A) = \{\Omega_{A_i} : \Omega_{A_i} \widetilde{\subseteq} \Omega_A, i \in I\}$$

and its cardinality is given by

$$|\widetilde{PW}(\Omega_A)| = 2^{\sum_{i \in \mathbb{N}} |X_i|},$$

where $|X_i|$ is the cardinality of the support set X_i of approximation image multi-set $\dddot{M}_i$ with respect to parameter $\ddot{e}_i$, where $i \in \mathbb{N}$.

Example 3.2. Let $H = [\frac{5}{a}, \frac{4}{b}, \frac{3}{c}]$, $E = \{\ddot{e}_1, \ddot{e}_2, \ddot{e}_3\}$, $A = \{\ddot{e}_1, \ddot{e}_2\} \subseteq E$ and a soft multi-set over H is

$$\Omega_A = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}.$$

Then $|\widetilde{PW}(\Omega_A)| = 2^{|X_1|+|X_2|} = 2^{2+2} = 2^4 = 16$, where $|X_1| = 2$, since $X_1 = \{a, b\}$ and $|X_2| = 2$, since $X_2 = \{b, c\}$.

The soft power whole multi-set of the soft multi-set Ω_A is given by $\widetilde{PW}(\Omega_A) = \{\Omega_{A_1}, \Omega_{A_2}, \cdots, \Omega_{A_{16}}\}$, where
$\Omega_{A_1} = \Omega_\emptyset$,
$\Omega_{A_2} = \{(\ddot{e}_1, [\frac{5}{a}])\}$,
$\Omega_{A_3} = \{(\ddot{e}_1, [\frac{4}{b}])\}$,
$\Omega_{A_4} = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}])\}$,
$\Omega_{A_5} = \{(\ddot{e}_2, [\frac{4}{b}])\}$,
$\Omega_{A_6} = \{(\ddot{e}_2, [\frac{3}{c}])\}$,
$\Omega_{A_7} = \{(\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}$,
$\Omega_{A_8} = \{(\ddot{e}_1, [\frac{5}{a}]), (\ddot{e}_2, [\frac{4}{b}])\}$,
$\Omega_{A_9} = \{(\ddot{e}_1, [\frac{5}{a}]), (\ddot{e}_2, [\frac{3}{c}])\}$,
$\Omega_{A_{10}} = \{(\ddot{e}_1, [\frac{5}{a}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}$,
$\Omega_{A_{11}} = \{(\ddot{e}_1, [\frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}])\}$,
$\Omega_{A_{12}} = \{(\ddot{e}_1, [\frac{4}{b}]), (\ddot{e}_2, [\frac{3}{c}])\}$,
$\Omega_{A_{13}} = \{(\ddot{e}_1, [\frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}$,
$\Omega_{A_{14}} = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}])\}$,
$\Omega_{A_{15}} = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{3}{c}])\}$,
$\Omega_{A_{16}} = \Omega_A$.

Example 3.3. Let $H = [\frac{1}{2}, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{2}{6}, \frac{5}{7}, \frac{1}{8}, \frac{5}{9}, \frac{4}{10}]$ and $E = \{\ddot{e}_1, \ddot{e}_2, \ddot{e}_3, \ddot{e}_4, \ddot{e}_5, \ddot{e}_6\}$ where
$\ddot{e}_1$ denotes divisibility by 2,
$\ddot{e}_2$ denotes divisibility by 3,
$\ddot{e}_3$ denotes divisibility by 4,
$\ddot{e}_4$ denotes divisibility by 5,
$\ddot{e}_5$ denotes divisibility by 6,
$\ddot{e}_6$ denotes divisibility by prime numbers.
Let $A = \{\ddot{e}_3, \ddot{e}_4, \ddot{e}_5\} \subseteq E$ and a soft multi-set over H is

$$\Omega_A = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}.$$

Then $|\widetilde{PW}(\Omega_A)| = 2^{|X_1|+|X_2|+|X_3|} = 2^{2+2+1} = 2^5 = 32$,
where $|X_1| = 2$, since $X_1 = \{4, 8\}$, $|X_2| = 2$, since $X_2 = \{5, 10\}$ and $|X_3| = 1$, since $X_3 = \{6\}$.
The soft power whole multi-set of the SMS Ω_A is given by $\widetilde{PW}(\Omega_A) = \{\Omega_{A_1}, \Omega_{A_2}, \cdots, \Omega_{A_{32}}\}$, where
$\Omega_{A_1} = \Omega_\emptyset$,
$\Omega_{A_2} = \{(\ddot{e}_3, [\frac{2}{4}])\}$,
$\Omega_{A_3} = \{(\ddot{e}_3, [\frac{1}{8}])\}$,
$\Omega_{A_4} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}])\}$,
$\Omega_{A_5} = \{(\ddot{e}_4, [\frac{3}{5}])\}$,
$\Omega_{A_6} = \{(\ddot{e}_4, [\frac{4}{10}])\}$,
$\Omega_{A_7} = \{(\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}])\}$,
$\Omega_{A_8} = \{(\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_9} = \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}])\}$,
$\Omega_{A_{10}} = \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{4}{10}])\}$,
$\Omega_{A_{11}} = \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}])\}$,
$\Omega_{A_{12}} = \{(\ddot{e}_3, [\frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}])\}$,

$\Omega_{A_{13}} = \{(\ddot{e}_3, [\frac{1}{8}]), (\ddot{e}_4, [\frac{4}{10}])\}$,
$\Omega_{A_{14}} = \{(\ddot{e}_3, [\frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}])\}$,
$\Omega_{A_{15}} = \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{16}} = \{(\ddot{e}_3, [\frac{1}{8}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{17}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{18}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}])\}$,
$\Omega_{A_{19}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{4}{10}])\}$,
$\Omega_{A_{20}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}])\}$,
$\Omega_{A_{21}} = \{(\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{22}} = \{(\ddot{e}_4, [\frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{23}} = \{(\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{24}} = \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{25}} = \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{26}} = \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{27}} = \{(\ddot{e}_3, [\frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{28}} = \{(\ddot{e}_3, [\frac{1}{8}]), (\ddot{e}_4, [\frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{29}} = \{(\ddot{e}_3, [\frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{30}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{31}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}$,
$\Omega_{A_{32}} = \Omega_A$.

Definition 3.4. "Let Ω_A be a soft multi-set over universal multi-set H. A SMS-topology on a soft multi-set Ω_A, denoted by $\widetilde{\tau}$, is a collection of soft multi subsets of Ω_A having the following properties:
(i) $\Omega_\emptyset, \Omega_A \in \widetilde{\tau}$.
(ii) Union of any number of members of $\widetilde{\tau}$ belongs to $\widetilde{\tau}$
i.e. $\{\Omega_{A_i} \widetilde{\subseteq} \Omega_A : i \in I \subseteq \mathbb{N}\} \widetilde{\subseteq} \widetilde{\tau} \Rightarrow \widetilde{\bigcup}_{i\in I} \Omega_{A_i} \in \widetilde{\tau}$.
(iii) Intersection of finite number of members of $\widetilde{\tau}$ belongs to $\widetilde{\tau}$
i.e. $\{\Omega_{A_i} \widetilde{\subseteq} \Omega_A : 1 \le i \le n, n \in \mathbb{N}\} \widetilde{\subseteq} \widetilde{\tau} \Rightarrow \widetilde{\bigcap}_{1\le i\le n} \Omega_{A_i} \in \widetilde{\tau}$.
Then a SMS topological space is denoted by $(\Omega_A, \widetilde{\tau})$" (Mukherjee *et al.* (2014), and Tokat and Osmanoglu (2011,2013)).

Example 3.5. Let $H = [\frac{5}{a}, \frac{4}{b}, \frac{3}{c}]$, $E = \{\ddot{e}_1, \ddot{e}_2, \ddot{e}_3\}$, $A = \{\ddot{e}_1, \ddot{e}_2\} \subseteq E$ and a soft multi-set over H is $\Omega_A = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}$ as given in Example 3.2. Then
$\widetilde{\tau_1} = \{\Omega_\emptyset, \Omega_A\}$, $\widetilde{\tau_2} = \widetilde{PW}(\Omega_A)$,
and $\widetilde{\tau_3} = \{\Omega_\emptyset, \{(\ddot{e}_1, [\frac{4}{b}])\}, \{(\ddot{e}_1, [\frac{4}{b}]), (\ddot{e}_2, [\frac{3}{c}])\}, \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}])\}, \Omega_A\}$
are three SMS topologies on the soft multi-set Ω_A.
Likewise $\widetilde{\tau_4} = \{\Omega_\emptyset, \{(\ddot{e}_1, [\frac{5}{a}])\}, \{(\ddot{e}_1, [\frac{4}{b}])\}, \Omega_A\}$ is not a SMS-topology on Ω_A.

Example 3.6. Take soft multi-set (SMS)

$$\Omega_A = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}.$$

which is same as given in Example 3.3. So that
$\widetilde{\tau_1} = \{\Omega_\emptyset, \Omega_A\}$,
$\widetilde{\tau_2} = \{\Omega_\emptyset, \Omega_{A_{24}}, \Omega_{A_{26}}, \Omega_{A_{30}}, \Omega_A\}$ or

$\widetilde{\tau_2} = \{\Omega_\emptyset, \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}, \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\},$
$\{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}, \Omega_A\}$ and
$\widetilde{\tau_3} = \widetilde{PW}(\Omega_A)$
are SMS topologies on the SMS Ω_A.

Throughout this work, we use the following definition of complement in a SMS topological space.

Definition 3.7. The soft multi complement $\Omega_B^{\tilde{c}}$ of a soft multi subset Ω_B in a SMS topological space $(\Omega_A, \widetilde{\tau})$ is defined as $\Omega_B^{\tilde{c}} = \Omega_A \widetilde{\setminus} \Omega_B$.

Definition 3.8. Let $\widetilde{\tau}$ be a SMS-topology then each of its element is called soft open multi-set (SOMS) and the complement of each soft open multi-set is called called a soft closed multi-set.

Example 3.9. Let $\widetilde{\tau_2}$ be the SMS-topology which considered in Example 3.6.
Since $\Omega_{A_{24}} = \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}$ is a soft open multi-set. Then $\Omega_{A_{24}}^{\tilde{c}} = \{(\ddot{e}_3, [\frac{1}{8}]), (\ddot{e}_4, [\frac{4}{10}])\}$ is a soft closed multi-set.

Remark. The union of two SMS-topologies on a SMS Ω_E may not be a SMS-topology on Ω_E.

Example 3.10. Let $H = [\frac{2}{g}, \frac{4}{h}, \frac{6}{i}]$, $E = \{\ddot{e}_1, \ddot{e}_2\}$, and $\widetilde{\tau_1} = \{\Omega_\emptyset, \Omega_{\widetilde{E}}, \Omega_{1_E}, \Omega_{2_E}, \Omega_{3_E}, \Omega_{4_E}\}$,
$\widetilde{\tau_2} = \{\Omega_\emptyset, \Omega_{\widetilde{E}}, \Omega_{5_E}, \Omega_{6_E}, \Omega_{7_E}, \Omega_{8_E}\}$ be two SMS topologies on $\Omega_{\widetilde{E}}$ where $\Omega_{1_E}, \Omega_{2_E}, \Omega_{3_E}, \Omega_{4_E}, \Omega_{5_E}, \Omega_{6_E}, \Omega_{7_E}$ and Ω_{8_E} are SMSs over H defined as follows:
$\Omega_{1_E} = \{(\ddot{e}_1, [\frac{4}{h}]), (\ddot{e}_2, [\frac{2}{g}])\}$,
$\Omega_{2_E} = \{(\ddot{e}_1, [\frac{4}{h}, \frac{6}{i}], (\ddot{e}_2, [\frac{2}{g}, \frac{4}{h}])\}$,
$\Omega_{3_E} = \{(\ddot{e}_1, [\frac{2}{g}, \frac{4}{h}]), (\ddot{e}_2, X)\}$,
$\Omega_{4_E} = \{(\ddot{e}_1, [\frac{2}{g}, \frac{4}{h}]), (\ddot{e}_2, [\frac{2}{g}, \frac{6}{i}])\}$,
$\Omega_{5_E} = \{(\ddot{e}_1, [\frac{4}{h}]), (\ddot{e}_2, [\frac{2}{g}])\}$,
$\Omega_{6_E} = \{(\ddot{e}_1, [\frac{4}{h}, \frac{6}{i}], (\ddot{e}_2, [\frac{2}{g}, \frac{4}{h}])\}$,
$\Omega_{7_E} = \{(\ddot{e}_1, [\frac{2}{g}, \frac{4}{h}]), (\ddot{e}_2, [\frac{2}{g}, \frac{4}{h}])\}$,
$\Omega_{8_E} = \{(\ddot{e}_1, [\frac{4}{h}]), (\ddot{e}_2, [\frac{2}{g}, \frac{6}{i}])\}$.
Now, we define
$\widetilde{\tau} = \widetilde{\tau_1} \widetilde{\cap} \widetilde{\tau_2}$
$= \{\Omega_{1_E}, \Omega_{2_E}, \Omega_{3_E}, \Omega_{4_E}, \Omega_{5_E}, \Omega_{6_E}, \Omega_{7_E}, \Omega_{8_E}\}$.
If we take $\Omega_{2_E} \widetilde{\cup} \Omega_{7_E} = H_E$. Then
$h_E(\ddot{e}_1) = f_{2_E}(\ddot{e}_1) \cup f_{7_E}(\ddot{e}_1) = [\frac{4}{h}, \frac{6}{i}] \cup [\frac{2}{g}, \frac{4}{h}] = H$
$h_E(\ddot{e}_2) = f_{2_E}(\ddot{e}_2) \cup f_{7_E}(\ddot{e}_2) = [\frac{2}{g}, \frac{4}{h}] \cup [\frac{2}{g}, \frac{4}{h}] = [\frac{2}{g}, \frac{4}{h}]$
but $H_E \notin \widetilde{\tau}$. Thus $\widetilde{\tau}$ is not a SMS-topology on $\Omega_{\widetilde{E}}$.

Definition 3.11. Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\widetilde{\mathcal{B}} \widetilde{\subseteq} \widetilde{\tau}$. If every element of $\widetilde{\tau}$ can be written as a union of members of $\widetilde{\mathcal{B}}$, then $\widetilde{\mathcal{B}}$ is called a soft multi basis for the SMS-topology $\widetilde{\tau}$.

Example 3.12. Let $H = [\frac{5}{a}, \frac{4}{b}, \frac{3}{c}]$, $E = \{\ddot{e}_1, \ddot{e}_2, \ddot{e}_3\}$, $A = \{\ddot{e}_1, \ddot{e}_2\} \subseteq E$ and a SMS over H is $\Omega_A = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}$. Let $\widetilde{\tau_2} = \widetilde{PW}(\Omega_A)$. Then $\widetilde{\mathcal{B}} = \{\Omega_\emptyset, \Omega_{A_2}, \Omega_{A_3}, \Omega_{A_5}, \Omega_{A_6}\}$ or

$$\widetilde{\mathcal{B}} = \{\Omega_\emptyset, \{(\ddot{e}_1, [\frac{5}{a}])\}, \{(\ddot{e}_1, [\frac{4}{b}])\}, \{(\ddot{e}_2, [\frac{4}{b}])\}, \{(\ddot{e}_2, [\frac{3}{c}])\}\}$$

is a soft multi basis for the SMS-topology $\widetilde{\tau_2}$.

Example 3.13. Consider the SMS-topology $\widetilde{\tau_3}$ that is given in Example 3.6. Since $\widetilde{\tau_3} = \widetilde{PW}(\Omega_A)$. Then $\widetilde{\mathcal{B}} = \{\Omega_\emptyset, \Omega_{A_2}, \Omega_{A_3}, \Omega_{A_5}, \Omega_{A_6}, \Omega_{A_8}\}$ is a soft multi basis for the SMS-topology $\widetilde{\tau_3}$.

Definition 3.14. Let $(\Omega_A, \widetilde{\tau_{\Omega_A}})$ be a SMS topological space and Ω_B is contained in Ω_A. Let $\widetilde{\tau_{\Omega_B}}$ be the collection of Ω_{B_i} such that $\Omega_{B_i} = \Omega_{A_i} \widetilde{\cap} \Omega_B$ where each Ω_{A_i} are contained in $\widetilde{\tau_{\Omega_A}}$.
Then $\widetilde{\tau_{\Omega_B}}$ is called a soft multi subspace topology or soft multi relative topology on Ω_B. Hence $(\Omega_B, \widetilde{\tau_{\Omega_B}})$ is soft multi subspace of $(\Omega_A, \widetilde{\tau_{\Omega_A}})$.

Theorem 3.15. *Let $(\Omega_A, \widetilde{\tau_{\Omega_A}})$ be a SMS topological space and $\Omega_B \widetilde{\subseteq} \Omega_A$. Then a soft multi subspace topology $\widetilde{\tau_{\Omega_B}}$ on Ω_B is a SMS-topology.*

Proof. (i) Since $\Omega_B \widetilde{\subseteq} \Omega_A$ and $\Omega_\phi \widetilde{\subseteq} \Omega_A$. Then clearly Ω_ϕ and Ω_B are contained in $\widetilde{\tau_{\Omega_B}}$ this is so because $\Omega_\phi \widetilde{\cap} \Omega_B = \Omega_\phi$ and $\Omega_A \widetilde{\cap} \Omega_B = \Omega_B$, where Ω_ϕ, Ω_A are in $\widetilde{\tau_{\Omega_A}}$.
(ii)-(iii) Since $\widetilde{\tau_{\Omega_A}}$ SMS topology, then by the given relations

$$\widetilde{\bigcap}_{i=1}^{n} (\Omega_{A_i} \widetilde{\cap} \Omega_B) = (\widetilde{\bigcap}_{i=1}^{n} \Omega_{A_i}) \widetilde{\cap} \Omega_B,$$

$$\widetilde{\bigcup}_{i \in I} (\Omega_{A_i} \widetilde{\cup} \Omega_B) = (\widetilde{\bigcap}_{i \in I} \Omega_{A_i}) \widetilde{\cap} \Omega_B$$

$\widetilde{\tau_{\Omega_B}}$ is the SMS topology on Ω_B. □

Example 3.16. Let us consider SMS-topology $\widetilde{\tau_3}$ on Ω_A as given in Example 3.5. Let $\Omega_B = \Omega_{A_{10}} = \{(\ddot{e}_1, [\frac{5}{a}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}$, and $\widetilde{\tau_3} = \{\Omega_\emptyset, \{(\ddot{e}_1, [\frac{4}{b}])\}, \{(\ddot{e}_1, [\frac{4}{b}]), (\ddot{e}_2, [\frac{3}{c}])\}, \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}])\}, \Omega_A\}$ then $\widetilde{\tau_{\Omega_B}} = \{\Omega_\emptyset, \Omega_{A_6}, \Omega_{A_9}, \Omega_{A_{10}}\}$. So $(\Omega_B, \widetilde{\tau_{\Omega_B}})$ is soft multi subspace of $(\Omega_A, \widetilde{\tau_3})$.

Example 3.17. Let us consider the SMS-topology $\widetilde{\tau_2}$ that is given in Example 3.6. Let $\Omega_B = \Omega_{A_{11}} = \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}])\}$ then $\widetilde{\tau_{\Omega_B}} = \{\Omega_\emptyset, \Omega_{A_9}, \Omega_{A_{11}}\}$. So $(\Omega_B, \widetilde{\tau_{\Omega_B}})$ is soft multi subspace of $(\Omega_A, \widetilde{\tau_2})$.

Definition 3.18. "Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B \widetilde{\subseteq} \Omega_A$. Then the soft multi interior of Ω_B, denoted by $Int(\Omega_B)$ or Ω_B°, is the soft multi union of all soft open multi subsets of Ω_B".

Example 3.19. Let us consider the SMS-topology $\widetilde{\tau_3}$ given in Example 3.5. If $\Omega_B = \Omega_{A_{13}} = \{(\ddot{e}_1, [\frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}$, then $\Omega_B^\circ = \Omega_\emptyset \widetilde{\cup} \Omega_{A_3} \widetilde{\cup} \Omega_{A_{12}} = \Omega_{A_{12}}$.

Example 3.20. Let us consider the SMS-topology $\widetilde{\tau_2}$ given in Example 3.6. If $\Omega_B = \Omega_{A_{17}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_5, [\frac{2}{6}])\}$, then $\Omega_B^\circ = \Omega_\emptyset$.

Definition 3.21. Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B \widetilde{\subseteq} \Omega_A$. Then the soft multi closure of Ω_B, denoted by $Cl(\Omega_B)$ or $\overline{\Omega}_B$, is the soft multi intersection of all soft closed super multi-sets of Ω_B.

Example 3.22. Let us consider the SMS-topology $\widetilde{\tau_3}$ given in Example 3.5. If $\Omega_B = \Omega_{A_{10}} = \{(\ddot{e}_1, [\frac{5}{a}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}$, then $\Omega_{A_3}^{\widetilde{c}} = \{(\ddot{e}_1, [\frac{5}{a}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\} = \Omega_B$ and $\Omega_\phi^{\widetilde{c}} = \Omega_A$ are soft closed super multi-sets of Ω_B. Hence $\overline{\Omega}_B = \Omega_A \widetilde{\cap} \Omega_B = \Omega_B$.

Example 3.23. Let us consider the SMS-topology $\widetilde{\tau}_2$ given in Example 3.6. If $\Omega_B = \Omega_{A_3} = \{(\ddot{e}_3, [\frac{1}{8}])\}$, then $\Omega^{\widetilde{c}}_{A_{24}} = \Omega_{A_{13}}, \Omega^{\widetilde{c}}_{A_{26}} = \Omega_{A_3}$ and $\Omega^{\widetilde{c}}_{\phi} = \Omega_A$ are soft closed super multi-sets of Ω_B. Hence $\overline{\Omega}_B = \Omega_{A_3}\widetilde{\cap}\Omega_{A_{13}}\widetilde{\cap}\Omega_A = \Omega_{A_3}$.

Theorem 3.24. *Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B, \Omega_C\widetilde{\subseteq}\Omega_A$. Then,*
(i) $(\Omega^{\circ}_B)^{\circ} = \Omega^{\circ}_B$
(ii) $\Omega_B\widetilde{\subseteq}\Omega_C \Rightarrow \Omega^{\circ}_B\widetilde{\subseteq}\Omega^{\circ}_C$
(iii) $\Omega^{\circ}_B\widetilde{\cap}\Omega^{\circ}_C = (\Omega_B\widetilde{\cap}\Omega_C)^{\circ}$
(iv) $\Omega^{\circ}_B\widetilde{\cup}\Omega^{\circ}_C\widetilde{\subseteq}(\Omega_B\widetilde{\cup}\Omega_C)^{\circ}$.
(v) $(\overline{\overline{\Omega_B}}) = \overline{\Omega_B}$
(vi) $\Omega_C\widetilde{\subseteq}\Omega_B \Rightarrow \overline{\Omega_C}\widetilde{\subseteq}\overline{\Omega_B}$
(vii) $\overline{(\Omega_B\widetilde{\cap}\Omega_C)}\widetilde{\subseteq}\overline{\Omega_B}\widetilde{\cap}\overline{\Omega_C}$
(viii) $\overline{(\Omega_B\widetilde{\cup}\Omega_C)} = \overline{\Omega_B}\widetilde{\cup}\overline{\Omega_C}$.
(ix) $\Omega^{\circ}_B\widetilde{\subseteq}\Omega_B\widetilde{\subseteq}\overline{\Omega_B}$

Proof. The proof follows by Definition 3.18 and Definition 3.21.

□

Example 3.25. Let $U = [\frac{2}{g}, \frac{4}{h}, \frac{6}{i}]$, $E = \{\ddot{e}_1, \ddot{e}_2\}$ and
$\widetilde{\tau} = \{\Omega_{\emptyset}, \Omega_{\widetilde{E}}, \Omega_{1_E}, \Omega_{2_E}, \Omega_{3_E}, ..., \Omega_{7_E}\}$, where
$\Omega_{1_E} = \{(\ddot{e}_1, [\frac{2}{g}, \frac{4}{h}]), (\ddot{e}_2, [\frac{2}{g}, \frac{4}{h}])\}$,
$\Omega_{2_E} = \{(\ddot{e}_1, [\frac{4}{h}]), (\ddot{e}_2, [\frac{2}{g}, \frac{6}{i}])\}$,
$\Omega_{3_E} = \{(\ddot{e}_1, [\frac{4}{h}, \frac{6}{i}]), (\ddot{e}_2, [\frac{2}{g}])\}$,
$\Omega_{4_E} = \{(\ddot{e}_1, [\frac{4}{h}]), (\ddot{e}_2, [\frac{2}{g}])\}$,
$\Omega_{5_E} = \{(\ddot{e}_1, [\frac{2}{g}, \frac{4}{h}]), (\ddot{e}_2, U)\}$,
$\Omega_{6_E} = \{(\ddot{e}_1, U), (\ddot{e}_2, [\frac{2}{g}, \frac{4}{h}])\}$,
$\Omega_{7_E} = \{(\ddot{e}_1, [\frac{4}{h}, \frac{6}{i}]), (\ddot{e}_2, [\frac{2}{g}, \frac{4}{h}])\}$.
Then $(\Omega_{\widetilde{E}}, \widetilde{\tau})$ is a soft multi-set topological space.
Let Ω_E and $\ddot{\Omega}_E$ are defined as follows:
$\Omega_E = \{(\ddot{e}_1, [\frac{2}{g}, \frac{6}{i}]), (\ddot{e}_2, \emptyset)\}$,
$\ddot{\Omega}_E = \{(\ddot{e}_1, [\frac{4}{h}, \frac{6}{i}]), (\ddot{e}_2, [\frac{2}{g}, \frac{4}{h}])\}$.
Then $\Omega_E\widetilde{\cap}\ddot{\Omega}_E = \{(\ddot{e}_1, [\frac{6}{i}]), (\ddot{e}_2, \emptyset)\}$.
Now, $\overline{\Omega_E} = \Omega_{\widetilde{E}}\widetilde{\cap}\Omega^{\widetilde{c}}_{2_E}\widetilde{\cap}\Omega^{\widetilde{c}}_{4_E} = \Omega^{\widetilde{c}}_{2_E}$ and $\overline{\ddot{\Omega}_E} = \Omega_{\widetilde{E}}$.
Therefore $\overline{\Omega_E}\widetilde{\cap}\overline{\ddot{\Omega}_E} = \overline{\Omega_E}$.
Also $\overline{\Omega_E\widetilde{\cap}\ddot{\Omega}_E} = \widetilde{\cap}\{\Omega_{\widetilde{E}}, \Omega^{\widetilde{c}}_{1_E}, \Omega^{\widetilde{c}}_{2_E}, \Omega^{\widetilde{c}}_{4_E}, \Omega^{\widetilde{c}}_{5_E}\} = \Omega^{\widetilde{c}}_{5_E}$.
So $\overline{\Omega_E\widetilde{\cap}\ddot{\Omega}_E}\widetilde{\subseteq}\overline{\Omega_E}\widetilde{\cap}\overline{\ddot{\Omega}_E}$ but $\overline{\Omega_E}\widetilde{\cap}\overline{\ddot{\Omega}_E}\widetilde{\nsubseteq}\overline{\Omega_E\widetilde{\cap}\ddot{\Omega}_E}$.
Hence, $\overline{\Omega_E\widetilde{\cap}\ddot{\Omega}_E} \neq \overline{\Omega_E}\widetilde{\cap}\overline{\ddot{\Omega}_E}$.

Definition 3.26. Let $(\Omega_A, \widetilde{\tau})$ be a soft multi-set topological space and $\Omega_B\widetilde{\subseteq}\Omega_A$. The soft multi interior of soft multi complement of Ω_B is called the soft multi exterior of Ω_B and is denoted by $Ext(\Omega_B)$ *or* $\Omega^{\widetilde{e}}_B$. In other words, $\Omega^{\widetilde{e}}_B = (\Omega^{\widetilde{c}}_B)^{\circ}$.

Example 3.27. From Example 3.5, we take SMS-topology $\widetilde{\tau}_3$. Then for

$\Omega_B = \Omega_{A_{14}} = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}])\}$, then $\Omega_{A_{14}}^{\widetilde{c}} = \{(\ddot{e}_2, [\frac{3}{c}])\} = \Omega_{A_6}$. Hence $\Omega_B^{\widetilde{e}} = (\Omega_B^{\widetilde{c}})^{\circ} = \Omega_{\phi}$, (because null soft multi-set is the only soft open multi subset contained in $\Omega_B^{\widetilde{c}}$).

Example 3.28. From Example 3.6, we take SMS-topology $\widetilde{\tau}_2$. Then for $\Omega_B = \Omega_{A_{30}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}$, then $\Omega_B^{\widetilde{c}} = \{(\ddot{e}_4, [\frac{4}{10}])\} = \Omega_{A_6}$. Hence $\Omega_B^{\widetilde{e}} = (\Omega_B^{\widetilde{c}})^{\circ} = \Omega_{\phi}$, (because null soft multi-set is the only soft open multi subset contained in $\Omega_B^{\widetilde{c}}$).

Theorem 3.29. *Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B, \Omega_C \widetilde{\subseteq} \Omega_A$. Then,*
(i) $(\Omega_B \widetilde{\cup} \Omega_C)^{\widetilde{e}} = (\Omega_B)^{\widetilde{e}} \widetilde{\cap} (\Omega_C)^{\widetilde{e}}$,
(ii) $(\Omega_B)^{\widetilde{e}} \widetilde{\cup} (\Omega_C)^{\widetilde{e}} \widetilde{\subseteq} (\Omega_B \widetilde{\cap} \Omega_C)^{\widetilde{e}}$.

Proof. (i) $(\Omega_B \widetilde{\cup} \Omega_C)^{\widetilde{e}} = ((\Omega_B \widetilde{\cup} \Omega_C)^{\widetilde{c}})^{\circ} = (\Omega_B^{\widetilde{c}} \widetilde{\cap} \Omega_C^{\widetilde{c}})^{\circ} = (\Omega_B^{\widetilde{c}})^{\circ} \widetilde{\cap} (\Omega_C^{\widetilde{c}})^{\circ} = (\Omega_B)^{\widetilde{e}} \widetilde{\cap} (\Omega_C)^{\widetilde{e}}$
(ii) $(\Omega_B)^{\widetilde{e}} \widetilde{\cup} (\Omega_C)^{\widetilde{e}} = (\Omega_B^{\widetilde{c}})^{\circ} \widetilde{\cup} (\Omega_C^{\widetilde{c}})^{\circ} \widetilde{\subseteq} (\Omega_B^{\widetilde{c}} \widetilde{\cup} \Omega_C^{\widetilde{c}})^{\circ} = (\Omega_B^{\circ} \widetilde{\cup} (\Omega_C^{\widetilde{c}})^{\circ} = (\Omega_B \widetilde{\cap} \Omega_C)^{\widetilde{e}}$. □

Definition 3.30. "Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space. A soft multi point $\alpha \in \Omega_A$ is said to be a soft multi interior point of the soft multi-set Ω_A if there is a soft open multi-set Ω_B such that $\alpha \in \Omega_B \widetilde{\subseteq} \Omega_A$.
Moreover, If α is soft multi interior point of the soft multi-set Ω_A then Ω_A is called soft multi neighborhood (or soft multi open neighborhood) of α. Thus $\widetilde{\nu}(\alpha) = \{\Omega_B : \Omega_B \in \widetilde{\tau}\}$ is the family of soft multi neighborhoods of α" (Mukherjee *et al.* (2014), and Tokat and Osmanoglu (2011,2013)).

Example 3.31. Let $\Omega_A = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}, \frac{3}{c}])\}$ be the soft multi-set as given in Example 3.5 and $\widetilde{\tau}_3 = \{\Omega_{\emptyset}, \{(\ddot{e}_1, [\frac{4}{b}])\}, \{(\ddot{e}_1, [\frac{4}{b}]), (\ddot{e}_2, [\frac{3}{c}])\}, \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}])\}, \Omega_A\}$ be a SMS-topology on the soft multi-set Ω_A.
Let $\alpha = (\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]) \in \Omega_A$ then $\alpha \in \Omega_{A_{14}} \widetilde{\subseteq} \Omega_A$, where $\Omega_{A_{14}} = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}])\}$ is the soft multi open neighborhood of α. Similarly $\alpha \in \Omega_A \widetilde{\subseteq} \Omega_A$ this shows that Ω_A is multi soft neighborhood of α. Thus $\widetilde{\nu}(\alpha) = \{\Omega_{A_{14}}, \Omega_A\}$ is the family of soft multi neighborhoods of α.

Example 3.32. Let $\Omega_A = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}$ be the soft multi-set as given in Example 3.6 and $\widetilde{\tau}_2 = \{\Omega_{\emptyset}, \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}, \{(\ddot{e}_3, [\frac{2}{4}]), (\ddot{e}_4, [\frac{3}{5}, \frac{4}{10}]), (\ddot{e}_5, [\frac{2}{6}])\}, \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}, \Omega_A\}$ be a SMS-topology on the SMS Ω_A. Let $\alpha = (\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]) \in \Omega_A$ then $\alpha \in \Omega_{A_{30}} \widetilde{\subseteq} \Omega_A$, where $\Omega_{A_{30}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}$ is the soft multi open neighborhood of α. Similarly $\alpha \in \Omega_A \widetilde{\subseteq} \Omega_A$ this shows that Ω_A is soft multi neighborhood of α. Thus $\widetilde{\nu}(\alpha) = \{\Omega_{A_{30}}, \Omega_A\}$ is the family of soft multi neighborhoods of α.

Theorem 3.33. *Let $\widetilde{\tau}$ be a SMS topology on SMS Ω_A. Then a subset Ω_B of Ω_A is said to be open if and only if it is neighborhood of each of its own soft multi point.*

Proof. Let Ω_B be soft multi open subset of Ω_A. Then for each soft multi point λ in Ω_B, we have $\lambda \widetilde{\in} \Omega_B \widetilde{\subseteq} \Omega_B$. This shows that Ω_B is a neighborhood of each of its own soft multi point.
Conversely, suppose that Ω_B is a neighborhood of each of its own soft multi point. Then for each soft multi point $\lambda \widetilde{\in} \Omega_B$ there exists soft multi open set Ω_{U_λ} such that $\lambda \widetilde{\in} \Omega_{U_\lambda} \widetilde{\subseteq} \Omega_B$.
This shows that $\Omega_B = \widetilde{\cup}\{\lambda\} \widetilde{\subseteq} \widetilde{\cup} \Omega_{U_\lambda} \widetilde{\subseteq} \Omega_B$.
Thus we get $\Omega_B = \widetilde{\subseteq} \widetilde{\cup} \Omega_{U_\lambda}$. This proves that Ω_B is soft multi open set.

□

Definition 3.34. "Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B \widetilde{\subseteq} \Omega_A$ and $\alpha \in \Omega_A$. If every multi soft neighborhood of α soft multi intersects Ω_B in some soft multi points other than α itself, then α is called a soft multi limit point of Ω_B. The collection of all soft multi limit points of Ω_B is denoted by Ω'_B.
In other words, if $(\Omega_A, \widetilde{\tau})$ is a SMS topological space and $\Omega_B \widetilde{\subseteq} \Omega_A$ and $\alpha \in \Omega_A$, then $\alpha \in \Omega'_B \Leftrightarrow \Omega_C \widetilde{\cap} (\Omega_B \widetilde{\setminus} \{\alpha\}) \neq \Omega_\phi$ for all $\Omega_C \in \widetilde{\nu}(\alpha)$".

Example 3.35. Consider example 3.31. If $\Omega_B = \Omega_{A_{14}}$ and $\alpha = (x_1, [\frac{5}{a}, \frac{4}{b}]) \in \Omega_A$, then $\alpha \in \Omega'_B$, since $\Omega_A \widetilde{\cap} (\Omega_B \widetilde{\setminus} \{\alpha\}) \neq \Omega_\phi$ and $\Omega_{A_{14}} \widetilde{\cap} (\Omega_B \widetilde{\setminus} \{\alpha\}) \neq \Omega_\phi$.

Example 3.36. Consider Example 3.32. If $\Omega_B = \Omega_{A_{30}}$ and $\alpha = (\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]) \in \Omega_A$, then $\alpha \in \Omega'_B$, since $\Omega_A \widetilde{\cap} (\Omega_B \widetilde{\setminus} \{\alpha\}) \neq \Omega_\phi$ and $\Omega_{A_{30}} \widetilde{\cap} (\Omega_B \widetilde{\setminus} \{\alpha\}) \neq \Omega_\phi$.

Theorem 3.37. *Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B \widetilde{\subseteq} \Omega_A$. Then, $\Omega_B \widetilde{\cup} \Omega'_B = \overline{\Omega_B}$.*

Proof. If $\alpha \in \Omega_B \widetilde{\cup} \Omega'_B$, then $\alpha \in \Omega_B$ or $\alpha \in \Omega'_B$. In this case, if $\alpha \in \Omega_B$, then $\alpha \in \overline{\Omega_B}$. If $\alpha \in \Omega'_B$, then $\Omega_C \widetilde{\cap} (\Omega_B \widetilde{\setminus} \{\alpha\}) \neq \Omega_\phi$ for all $\Omega_C \in \widetilde{\nu}(\alpha)$, and so $\Omega_C \widetilde{\cap} \Omega_B \neq \Omega_\phi$ for all $\Omega_C \in \widetilde{\nu}(\alpha)$; hence, $\alpha \in \overline{\Omega_B}$. Conversely, if $\alpha \in \overline{\Omega_B}$, then $\alpha \in \Omega_B$ or $\alpha \in \Omega'_B$. In this case, if $\alpha \in \Omega_B$, it is trivial that $\alpha \in \Omega_B \widetilde{\cup} \Omega'_B$. If $\alpha \notin \Omega_B$, then $\Omega_C \widetilde{\cap} (\Omega_B \widetilde{\setminus} \{\alpha\}) \neq \Omega_\phi$ for all $\Omega_C \in \widetilde{\nu}(\alpha)$. Therefore, $\alpha \in \Omega'_B$, so $\alpha \in \Omega_B \widetilde{\cup} \Omega'_B$. Hence $\Omega_B \widetilde{\cup} \Omega'_B = \overline{\Omega_B}$. □

Theorem 3.38. *Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B \widetilde{\subseteq} \Omega_A$. Then, Ω_B is soft closed multi-set if and only if $\Omega'_B \widetilde{\subseteq} \Omega_B$.*

Proof. $\overline{\Omega_B} = \Omega_B \Leftrightarrow \Omega_B \widetilde{\cup} \Omega'_B = \Omega_B \Leftrightarrow \Omega'_B \widetilde{\subseteq} \Omega_B$. □

Theorem 3.39. *Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B, \Omega_C \widetilde{\subseteq} \Omega_A$. Then,*
(i) $\Omega'_B \widetilde{\subseteq} \Omega_B$
(ii) $\Omega_B \widetilde{\subseteq} \Omega_C \Rightarrow \Omega'_B \widetilde{\subseteq} \Omega'_C$
(iii) $(\Omega_B \widetilde{\cap} \Omega_C)' \widetilde{\subseteq} \Omega'_B \widetilde{\cap} \Omega'_C$
(iv) $(\Omega_B \widetilde{\cup} \Omega_C)' = \Omega'_B \widetilde{\cup} \Omega'_C$
(v) Ω_B *is a soft closed multi-set* $\Leftrightarrow \Omega'_B \widetilde{\subseteq} \Omega_B$.

Proof. The proof is straightforward. □

Theorem 3.40. *Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B, \Omega_C \widetilde{\subseteq} \Omega_A$. Then,*
(i) $\overline{(\Omega_B^{\widetilde{c}})} = (\Omega_B^{\circ})^{\widetilde{c}}$
(ii) $(\overline{\Omega}_B)^{\widetilde{c}} = (\Omega_B^{\widetilde{c}})^{\circ}$
(iii) $\Omega_B^{\circ} = (\overline{(\Omega_B^{\widetilde{c}})})^{\widetilde{c}}$
(iv) $\overline{\Omega_B} = ((\Omega_B^{\widetilde{c}})^{\circ})^{\widetilde{c}}$
(v) $(\Omega_B \widetilde{\setminus} \Omega_C)^{\circ} \widetilde{\subseteq} \Omega_B{}^{\circ} \widetilde{\setminus} \Omega_C{}^{\circ}$.

Proof. (*i*) Let $\alpha \in \Omega_B$ such that $\alpha \notin \Omega_B^{\circ}$. Then, for each soft multi open neighborhood of Ω_C of α, Ω_C soft multi intersects $\Omega_B^{\widetilde{c}}$. Otherwise, for some soft multi open neighborhood Ω_C of α, $\Omega_C \widetilde{\cap} \Omega_B^{\widetilde{c}} = \Omega_\phi$ or $\Omega_C \widetilde{\subseteq} \Omega_B$. Since Ω_B° is the largest soft open multi-set in Ω_B, therefore $\alpha \in \Omega_C \widetilde{\subseteq} \Omega_B^{\circ}$, which is a contradiction. Therefore, $\alpha \in \overline{(\Omega_B^{\widetilde{c}})}$. Hence, $(\Omega_B^{\circ})^{\widetilde{c}} \widetilde{\subseteq} \overline{(\Omega_B^{\widetilde{c}})}$.

Conversely, suppose $\alpha \in \overline{\Omega_B^{\tilde{c}}}$, then by Definition 3.34, $\alpha \in \Omega_B^{\tilde{c}}$ or α is a soft multi limit point of $\Omega_B^{\tilde{c}}$. If $\alpha \in \Omega_B^{\tilde{c}}$, then $\alpha \in (\Omega_B^{\circ})^{\tilde{c}}$. In the second case, $\alpha \notin \Omega_B^{\circ}$. Otherwise, by the definition of soft multi limit point, $\Omega_B^{\circ}\widetilde{\cap}\Omega_B^{\tilde{c}} \neq \Omega_\phi$, which is false. Therefore, $\overline{(\Omega_B^{\tilde{c}})}\widetilde{\subseteq}(\Omega_B^{\circ})^{\tilde{c}}$.
Combining, we get (i).
(ii) Clearly
$(\overline{\Omega_B})^{\tilde{c}} = (\bigcap_{\Omega_{A_i}\widetilde{\supseteq}\Omega_B, \Omega_{A_i}^{\tilde{c}}\in\tilde{\tau}}\Omega_{A_i})^{\tilde{c}} = \widetilde{\bigcup}\Omega_{A_i}^{\tilde{c}} = (\Omega_B^{\tilde{c}})^{\circ}$.
(iii) and (iv) are directly obtained by taking the complements of (i) and (ii), respectively.
(v) $(\Omega_B\widetilde{\setminus}\Omega_C)^{\circ} = (\Omega_B\widetilde{\cap}\Omega_C^{\tilde{c}})^{\circ} = \Omega_B^{\circ}\widetilde{\cap}(\Omega_C^{\tilde{c}})^{\circ} = \Omega_B^{\circ}\widetilde{\cap}(\overline{\Omega_C})^{\tilde{c}}\ \widetilde{\subseteq}\Omega_B^{\circ}\widetilde{\cap}(\Omega_C^{\circ})^{\tilde{c}} = \Omega_B^{\circ}\widetilde{\setminus}\Omega_C^{\circ}$. □

Definition 3.41. Let $(\Omega_A, \tilde{\tau})$ be a SMS topological space and $\Omega_B\widetilde{\subseteq}\Omega_A$. The soft multi frontier or boundary of Ω_B is denoted by $\Omega_r(\Omega_B)$ or $\Omega_B^{\tilde{b}}$ and is defined as $\Omega_B^{\tilde{b}} = \overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\tilde{c}}}$. Stated differently, the soft multi points that do not belong to soft multi interior and exterior of Ω_B are in $\Omega_B^{\tilde{b}}$.

Example 3.42. From Example 3.5, we take SMS-topology $\tilde{\tau}_3$, then for $\Omega_B = \Omega_{A_{14}} = \{(\ddot{e}_1, [\frac{5}{a}, \frac{4}{b}]), (\ddot{e}_2, [\frac{4}{b}])\}$, then $\Omega_{A_{14}}^{\tilde{c}} = \{(\ddot{e}_2, [\frac{3}{c}])\} = \Omega_{A_6}$. Hence $\Omega_B^{\tilde{b}} = \overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\tilde{c}}} = \Omega_A\widetilde{\cap}\Omega_{A_6} = \Omega_{A_6}$.

Example 3.43. Let us consider the SMS-topology $\tilde{\tau}_2$ given in Example 3.6.
If $\Omega_B = \Omega_{A_{30}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}$, then $\Omega_{A_{30}}^{\tilde{c}} = \{(\ddot{e}_4, [\frac{4}{10}])\} = \Omega_{A_6}$. Hence $\Omega_B^{\tilde{b}} = \overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\tilde{c}}} = \Omega_A\widetilde{\cap}\Omega_{A_6} = \Omega_{A_6}$.

Theorem 3.44. *Let $(\Omega_A, \tilde{\tau})$ be a SMS topological space and $\Omega_B, \Omega_C\widetilde{\subseteq}\Omega_A$. Then,*
(i) $\Omega_B^{\tilde{b}}\widetilde{\subseteq}\overline{\Omega_B}$
(ii) $\Omega_B^{\tilde{b}} = (\Omega_B^{\tilde{c}})^{\tilde{b}}$
(iii) $\Omega_B^{\tilde{b}} = \overline{\Omega_B}\widetilde{\setminus}\Omega_B^{\circ}$.

Proof. (i) The proof is clear by definition of a soft multi boundary.
(ii) Take as given $\alpha \in \Omega_B^{\tilde{b}} \Leftrightarrow \Omega_C\widetilde{\cap}\Omega_B \neq \Omega_\phi$ and $\Omega_C\widetilde{\cap}{\Omega_B}^{\tilde{c}} \neq \Omega_\phi$ for all $\Omega_C \in \tilde{\nu}(\alpha) \Leftrightarrow \Omega_C\widetilde{\cap}{\Omega_B}^{\tilde{c}} \neq \Omega_\phi$ and $\Omega_C\widetilde{\cap}({\Omega_B}^{\tilde{c}})^{\tilde{c}} \neq \Omega_\phi$ for all $\Omega_C \in \tilde{\nu}(\alpha)$. Hence $\Omega_B^{\tilde{b}} = (\Omega_B^{\tilde{c}})^{\tilde{b}}$.
(iii) By using the definitions of a soft multi closure and a multi soft interior, we have
$\overline{\Omega_B}\widetilde{\setminus}\Omega_B^{\circ} = \overline{\Omega_B}\widetilde{\cap}(\Omega_B^{\circ})^{\tilde{c}} = \overline{\Omega_B}\widetilde{\cap}(\widetilde{\bigcup}_{\Omega_{B_i}\widetilde{\subseteq}\Omega_B, \Omega_{B_i}\in\tilde{\tau}}\Omega_{B_i})^{\tilde{c}} = \overline{\Omega_B}\widetilde{\cap}(\widetilde{\bigcap}{\Omega_{B_i}}^{\tilde{c}}) = \overline{\Omega_B}\widetilde{\cap}(\overline{{\Omega_{B_i}}^{\tilde{c}}}) = \Omega_B^{\tilde{b}}$. □

Theorem 3.45. *Let $(\Omega_A, \tilde{\tau})$ be a SMS topological space and $\Omega_B\widetilde{\subseteq}\Omega_A$. Then,*
(i) $(\Omega_B^{\tilde{b}})^{\tilde{c}} = \Omega_B^{\circ}\widetilde{\cup}(\Omega_B^{\tilde{c}})^{\circ} = \Omega_B^{\circ}\widetilde{\cup}\Omega_B^{\tilde{e}}$
(ii) $\overline{\Omega_B} = \Omega_B\widetilde{\cup}\Omega_B^{\tilde{b}}$
(iii) $\Omega_B^{\circ} = \Omega_B\widetilde{\setminus}\Omega_B^{\tilde{b}}$.

Proof. (i) $\Omega_B^{\circ}\widetilde{\cup}(\Omega_B^{\tilde{c}})^{\circ} = ((\Omega_B^{\circ})^{\tilde{c}})^{\tilde{c}}\widetilde{\cup}(((\Omega_B^{\tilde{c}})^{\circ})^{\tilde{c}})^{\tilde{c}} = [(\Omega_B^{\circ})^{\tilde{c}}\widetilde{\cap}((\Omega_B^{\tilde{c}})^{\circ})^{\tilde{c}}\ ]^{\tilde{c}} = [\overline{\Omega_B^{\tilde{c}}}\widetilde{\cap}\overline{\Omega_B}\ ]^{\tilde{c}} = (\Omega_B^{\tilde{b}})^{\tilde{c}}$.
(ii) $\Omega_B\widetilde{\cup}\Omega_B^{\tilde{b}} = \Omega_B\widetilde{\cup}(\overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\tilde{c}}}) = [\Omega_B\widetilde{\cup}\overline{\Omega_B}\ ]\widetilde{\cap}[\Omega_B\widetilde{\cup}\overline{\Omega_B^{\tilde{c}}}\ ] = \overline{\Omega_B}\widetilde{\cap}[\Omega_B\widetilde{\cup}\overline{\Omega_B^{\tilde{c}}}\ ] = \overline{\Omega_B}\widetilde{\cap}\Omega_A = \overline{\Omega_B}$.
(iii) $\Omega_B\widetilde{\setminus}\Omega_B^{\tilde{b}} = \Omega_B\widetilde{\cap}(\Omega_B^{\tilde{b}})^{\tilde{c}} = \Omega_B\widetilde{\cap}(\Omega_B^{\circ}\widetilde{\cup}(\Omega_B^{\tilde{c}})^{\circ})$ (by (i)) $= [\Omega_B\widetilde{\cap}\Omega_B^{\circ}]\widetilde{\cup}[\Omega_B\widetilde{\cap}(\Omega_B^{\tilde{c}})^{\circ}] = \Omega_B^{\circ}\widetilde{\cup}\Omega_\phi = \Omega_B^{\circ}$. □

Remark. From Theorem 3.45, it follows that $\Omega_A = \Omega_B^{\circ}\widetilde{\cup}\Omega_B^{\tilde{e}}\widetilde{\cup}\Omega_B^{\tilde{b}}$.

Theorem 3.46. *Let $(\Omega_A, \tilde{\tau})$ be a SMS topological space and $\Omega_B\widetilde{\subseteq}\Omega_A$. Then,*
(i) $\Omega_B^{\tilde{b}}\widetilde{\cap}\Omega_B^{\circ} = \Omega_\phi$
(ii) $\Omega_B^{\tilde{b}}\widetilde{\cap}\Omega_B^{\tilde{e}} = \Omega_\phi$.

Proof. (i) $\Omega_B^{\widetilde{b}}\widetilde{\cap}\Omega_B^{\circ} = (\overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\widetilde{c}}})\widetilde{\cap}\Omega_B^{\circ} = \overline{\Omega_B}\widetilde{\cap}(\Omega_B^{\circ})^{\widetilde{c}}\widetilde{\cap}\Omega_B^{\circ} = \Omega_\phi$.
(ii) $\Omega_B^{\widetilde{b}}\widetilde{\cap}\Omega_B^{\widetilde{e}} = (\Omega_B^{\widetilde{c}})^{\circ}\widetilde{\cap}(\overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\widetilde{c}}}) = (\Omega_B^{\widetilde{c}})^{\circ}\widetilde{\cap}\overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\widetilde{c}}} = (\overline{\Omega_B})^{\widetilde{c}}\widetilde{\cap}\overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\widetilde{c}}} = \Omega_\phi$. □

Theorem 3.47. *Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B\widetilde{\subseteq}\Omega_A$. Then,*
(i) Ω_B *is soft open multi-set* $\Leftrightarrow \Omega_B\widetilde{\cap}\Omega_B^{\widetilde{b}} = \Omega_\phi$
(ii) Ω_B *is soft closed multi-set* $\Leftrightarrow \Omega_B^{\widetilde{b}}\widetilde{\subseteq}\Omega_B$.
(iii) Ω_B *is both soft open multi-set and soft closed multi-set* $\Leftrightarrow \Omega_B^{\widetilde{b}} = \emptyset$.

Proof. (i) Let Ω_B is soft open multi-set. Then $\Omega_B^{\circ} = \Omega_B$. Thus $\Omega_B\widetilde{\cap}\Omega_B^{\widetilde{b}} = \Omega_B^{\circ}\widetilde{\cap}\Omega_B^{\widetilde{b}} = \Omega_\phi$ (by Theorem 3.46(i)).
Conversely, let $\Omega_B\widetilde{\cap}\overline{\Omega_B} = \Omega_\phi$. Then, $\Omega_B\widetilde{\cap}[\overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\widetilde{c}}}] = \Omega_\phi$, $\Omega_B\widetilde{\cap}\overline{\Omega_B^{\widetilde{c}}} = \Omega_\phi$, or $\overline{\Omega_B^{\widetilde{c}}}\widetilde{\subseteq}\Omega_B^{\widetilde{c}}$, which implies that $\Omega_B^{\widetilde{c}}$ is soft closed multi-set and hence, Ω_B is soft open multi-set.
(ii) Let Ω_B is soft closed multi-set. Then $\overline{\Omega_B} = \Omega_B$. Now, $\Omega_B^{\widetilde{b}} = \overline{\Omega_B}\widetilde{\cap}\overline{\Omega_B^{\widetilde{c}}}\widetilde{\subseteq}\overline{\Omega_B} = \Omega_B$, or $\Omega_B^{\widetilde{b}}\widetilde{\subseteq}\Omega_B$ and conversely.
(iii) We know that Ω_B is open $\Leftrightarrow (\Omega_B)^{\circ} = \Omega_B$ and Ω_B is closed $\Leftrightarrow \overline{\Omega_B} = \Omega_B$. Also by Theorem 3.45, we obtain $\overline{\Omega_B} = \Omega_B\widetilde{\cup}\Omega_B^{\widetilde{b}}$ and $\Omega_B^{\circ} = \Omega_B\widetilde{\setminus}\Omega_B^{\widetilde{b}}$. This completes the proof. □

Theorem 3.48. *Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B, \Omega_C\widetilde{\subseteq}\Omega_A$. Then,*
(i) $[\Omega_B\widetilde{\cup}\Omega_C]^{\widetilde{b}}\widetilde{\subseteq}[\Omega_B^{\widetilde{b}}\widetilde{\cap}\overline{\Omega_C^{\widetilde{c}}}]\widetilde{\cup}[\Omega_C^{\widetilde{b}}\widetilde{\cap}\overline{\Omega_B^{\widetilde{c}}}]$
(ii) $[\Omega_B\widetilde{\cap}\Omega_C]^{\widetilde{b}}\widetilde{\subseteq}[\Omega_B^{\widetilde{b}}\widetilde{\cap}\overline{\Omega_C}\ ]\widetilde{\cup}[\Omega_C^{\widetilde{b}}\widetilde{\cap}\overline{\Omega_B}\ ]$.

Proof. Proof is obvious. □

Theorem 3.49. *Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B\widetilde{\subseteq}\Omega_A$. Then,*
$((\Omega_B^{\widetilde{b}})^{\widetilde{b}})^{\widetilde{b}} = (\Omega_B^{\widetilde{b}})^{\widetilde{b}}$.

Proof. (i) $((\Omega_B^{\widetilde{b}})^{\widetilde{b}})^{\widetilde{b}} = \overline{(\Omega_B^{\widetilde{b}})^{\widetilde{b}}}\widetilde{\cap}\overline{((\Omega_B^{\widetilde{b}})^{\widetilde{b}})^{\widetilde{c}}} = (\Omega_B^{\widetilde{b}})^{\widetilde{b}}\widetilde{\cap}\overline{((\Omega_B^{\widetilde{b}})^{\widetilde{b}})^{\widetilde{c}}}$ (1)
Now, consider $((\Omega_B^{\widetilde{b}})^{\widetilde{b}})^{\widetilde{c}} = [\overline{(\Omega_B^{\widetilde{b}})}\widetilde{\cap}\overline{(\Omega_B^{\widetilde{b}})^{\widetilde{c}}}\ ]^{\widetilde{c}} = (\Omega_B^{\widetilde{b}}\widetilde{\cap}\overline{(\Omega_B^{\widetilde{b}})^{\widetilde{c}}})^{\widetilde{c}} = (\Omega_B^{\widetilde{b}})^{\widetilde{c}}\widetilde{\cup}(\overline{(\Omega_B^{\widetilde{b}})^{\widetilde{c}}})^{\widetilde{c}}$.
Therefore,
$\overline{(((\Omega_B^{\widetilde{b}})^{\widetilde{b}})^{\widetilde{c}})} = \overline{[(\Omega_B^{\widetilde{b}})^{\widetilde{c}}\widetilde{\cup}(\overline{(\Omega_B^{\widetilde{b}})^{\widetilde{c}}}\ )^{\widetilde{c}}\ ]} = \overline{((\Omega_B^{\widetilde{b}})^{\widetilde{c}}\ )}\widetilde{\cup}\overline{((\overline{(\Omega_B^{\widetilde{b}})^{\widetilde{c}}})^{\widetilde{c}})} = \Omega_C\widetilde{\cup}\overline{((\Omega_C)^{\widetilde{c}})} = \Omega_A$ (2)
where $\Omega_C = \overline{((\Omega_B^{\widetilde{b}})^{\widetilde{c}}\ )}$.
From (1) and (2), we have $((\Omega_B^{\widetilde{b}})^{\widetilde{b}})^{\widetilde{b}} = (\Omega_B^{\widetilde{b}})^{\widetilde{b}}\widetilde{\cap}\Omega_A = (\Omega_B^{\widetilde{b}})^{\widetilde{b}}$. □

Definition 3.50. Let $(\Omega_A, \widetilde{\tau})$ be a SMS topological space and $\Omega_B\widetilde{\subseteq}\Omega_A$. Then Ω_B is said to be a soft clopen multi-set if Ω_B is both soft open and soft closed multi-set.

Example 3.51. Since Ω_ϕ and Ω_A are always present in $\widetilde{\tau}$, so Ω_ϕ and Ω_A are soft open multi-sets. Moreover, Ω_ϕ and Ω_A are also soft closed multi-sets since $\Omega_\phi^{\widetilde{c}} = \Omega_A$ and $\Omega_A^{\widetilde{c}} = \Omega_\phi$. In fact, these two soft multi-sets are soft open and soft closed multi-sets simultaneously. Hence, Ω_ϕ and Ω_A are soft clopen multi-sets.

Example 3.52. Let us consider the SMS-topology $\widetilde{\tau}_3$ given in Example 3.6. Let $\Omega_B, \Omega_C\widetilde{\in}\widetilde{\tau}_3$, where $\Omega_B = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\}$, and $\Omega_C = \{(\ddot{e}_4, [\frac{4}{10}])\}$.
Then $\Omega_C^{\widetilde{c}} = \{(\ddot{e}_3, [\frac{2}{4}, \frac{1}{8}]), (\ddot{e}_4, [\frac{3}{5}]), (\ddot{e}_5, [\frac{2}{6}])\} = \Omega_B$. Hence Ω_B is a soft clopen multi-set.

Theorem 3.53. *Let* $(\Omega_A, \widetilde{\tau})$ *be a SMS topological space and* $\Omega_B \widetilde{\subseteq} \Omega_A$. $\Omega_B^{\widetilde{b}} = \Omega_\phi$ *if and only if* Ω_B *is soft clopen multi-set.*

Proof. Suppose that $\Omega_B^{\widetilde{b}} = \Omega_\phi$. First we prove that Ω_B is a soft closed multi-set. Consider

$$\Omega_B^{\widetilde{b}} = \Omega_\phi \Rightarrow \overline{\Omega_B} \widetilde{\cap} \overline{(\Omega_B^{\widetilde{c}})} = \Omega_\phi \Rightarrow \overline{\Omega_B} \widetilde{\subseteq} \overline{((\Omega_B^{\widetilde{c}})^{\widetilde{c}})} = \Omega_B^\circ \widetilde{\subseteq} \Omega_B \Rightarrow \overline{\Omega_B} \widetilde{\subseteq} \Omega_B \Rightarrow \overline{\Omega_B} = \Omega_B.$$

This implies that Ω_B is a soft closed multi-set. Now we now prove that Ω_B is a soft open multi-set. Consider

$$\Omega_B^{\widetilde{b}} = \Omega_\phi \Rightarrow \overline{\Omega_B} \widetilde{\cap} \overline{(\Omega_B^{\widetilde{c}})} = \Omega_\phi \text{ or } \Omega_B \widetilde{\cap} (\Omega_B^\circ)^{\widetilde{c}} = \Omega_\phi \Rightarrow \Omega_B \widetilde{\subseteq} \Omega_B^\circ$$

$\Omega_B^\circ = \Omega_B$. This implies that Ω_B is a soft open multi-set. Conversely, suppose that Ω_B is a soft clopen multi-set. Then,

$$\Omega_B^{\widetilde{b}} = \overline{\Omega_B} \widetilde{\cap} \overline{(\Omega_B^{\widetilde{c}})} = \overline{\Omega_B} \widetilde{\cap} (\Omega_B^\circ)^{\widetilde{c}} = \Omega_B \widetilde{\cap} \Omega_B^{\widetilde{c}} = \Omega_\phi.$$

□

4 MCDM based on SMS-topology

There are different kinds of decision-making methods for selection of a best alternative. Sometimes it is quite difficult to select an appropriate decision-making method with similar situation in our real life problems. However, MCDM method based on SMS-topology plays a enthusiastic role in our daily life and this is very helpful in selection of a best alternative. MCDM is the thought process of selecting a logical choice from the available options. The concept of aggregation operators in the framework of soft sets and fuzzy soft sets have been introduced by Çağman *et al.* (2011). We used the notion of aggregation operators to compute aggregate fuzzy soft sets and aggregate multi-sets.

4.1 MCDM for selection of best alternative of biopesticides

A big challenge to the agricultural department is to enlarge the production and to meet the demands of the increasing world population without destroying the environment. In modern agricultural exercises, the check of pests is generally completed by means of the extreme usage of agrochemicals, which is source of ambient pollution and the improvement of repellent pests. But biopesticides can proffer a best substitute to synthetic pesticides empowering safer check of pest communities. It is always a challenging task for a farmer to choose a best agrochemicals for biopesticides. Every farmer has to face many difficulties to save his fields from pests. For these challenging tasks various components are take into examination by the farmer either searching for agrochemicals in order to provide safety from pests attack, improve the soil quality, increase the quantity of crops, enhance the quality of crops. Major components of biopesticides include microbial pesticides, biochemical pesticides and biological control agent. The examples of biopesticides include insects, virus, bacteria, fungi, protozoan, and nematodes. Table 1 gives the comparison of merits and demerits of biopesticides and chemicals.

Biopesticides	Chemicals pesticides
Environmentally intelligent farming	Conflicting to intelligent farming
Cheaper, affordable	Costly, expansive
Warmly to non-target genus	Dangerous to non-target genus
Do not cause pollution	Serious pollution to the environment
Pests never develop resistance	Pests eventually become resistance
Expanding market inclination	Reduce market inclination
Fight their intended pests	End up affecting non target species
Derived from living organisms	Contain non-living organism

Table 1: Comparison analysis of biopesticides and chemicals

Algorithm 1 The selection of best alternative for biopesticides

Step 1: Input a suitable parameter set S and universal multi-set H.
Step 2: Input SMSs Ω_A and Ω_B over H.
Step 3: Construct SMS-topology $\widehat{\tau}$ containing Ω_A and Ω_B as soft open MSs in $\widehat{\tau}$.
Step 4: Compute the aggregate fuzzy soft sets by using the formula,
$\Gamma_A = \{(\mu_i, \Gamma_A(\mu_i)) : \mu_i \in S\}$, where $\Gamma_A(\mu_i) = \{\frac{k_i/|\Omega_A(\mu_i)|}{\omega_i} : \frac{k_i}{\omega_i} \in \Omega_A(\mu_i)\}$.
Step 5: Find resultant fuzzy soft set $\Gamma_A \vee \Gamma_B = \Gamma_{A\times B}$ by applying 'OR' operation on Γ_A and Γ_B.
Step 6: Use comparison table of $\Gamma_A \vee \Gamma_B$ to calculate row-sum (r_i) and column-sum (t_i) for ω_i, $\forall\ i$.
Step 7: Calculate the resulting score R_i of ω_i, $\forall\ i$.
Step 8: Optimal choice is ω_j that has $\max\{R_i\}$.
Step 9: Compute the SMS boundary of soft open multi-sets.
Step 10: Here non-null SMS boundary of SMS that contains $\frac{k_j}{\omega_j}$ is a decision set.

Figure 1 shows a brief flow-chart of Algorithm 1.

Assume that a farmer wants to safe his fields from pests by using leading alternative of biopesticides without damaging the sustainability of environment.
Let $H = [\frac{30}{\omega_1}, \frac{25}{\omega_2}, \frac{28}{\omega_3}, \frac{30}{\omega_4}]$ be the universe of some plants, where
ω_1 = Sheesham (Dalbergia sissoo),
ω_2 = Safeda (Eucalyptus),
ω_3 = Sukh Chain (Pongamia pinnata),
ω_4 = Neem (Azadirachta indica)
and the multiplicity of ω_i $(i = 1, 2, 3, 4)$ denotes the number of plants corresponding to ω_i. Consider the set of attributes $S = \{\mu_1, \mu_2, \mu_3, \mu_4\}$, where
μ_1 = provide safety from pests attack,
μ_2 = improve the soil quality,
μ_3 = increase the quantity of crops,

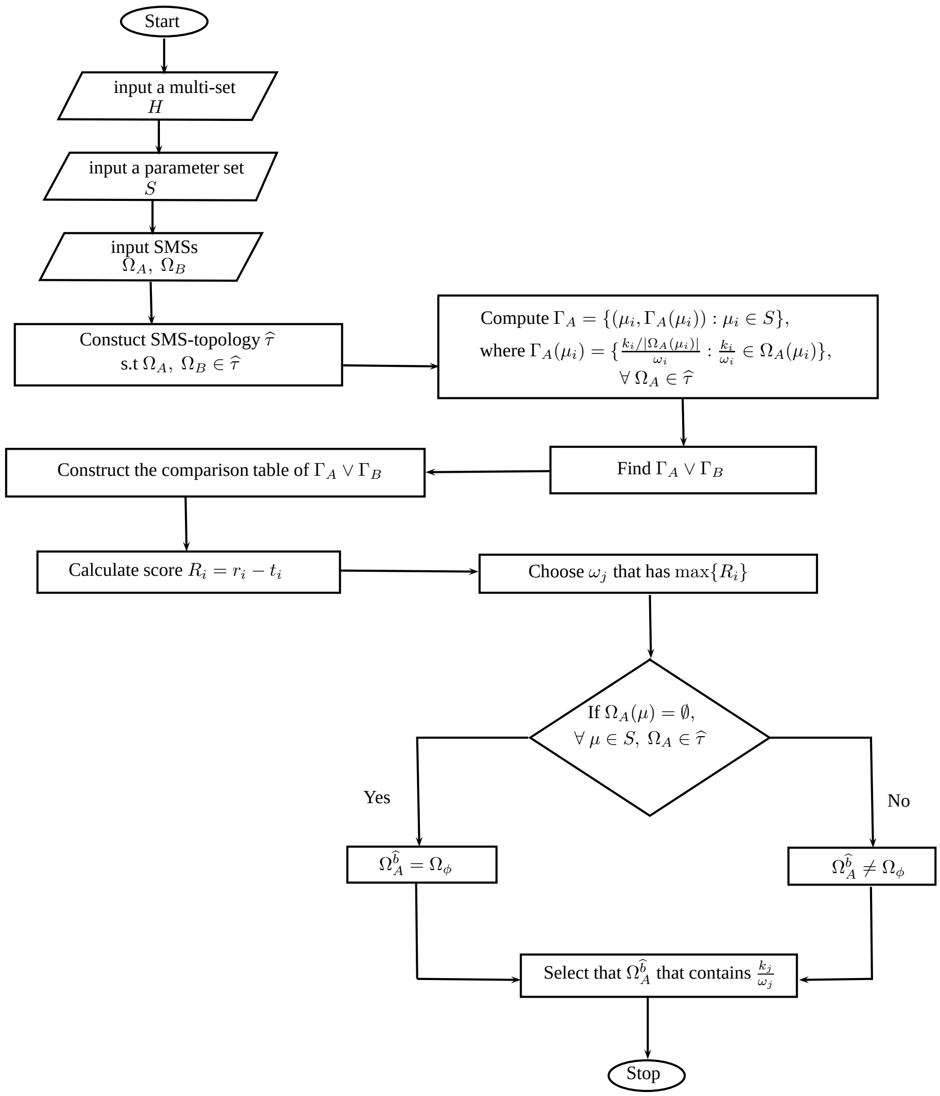

Figure 1: Graphical representation of Algorithm 1

$\mu_4 =$ enhance the quality of crops.
We here use the following algorithm to choose the best alternative of agrochemicals for biopesticides without damaging the environment to safe the fields from pests.

Two decision makers (DMs) Ω_1 and Ω_2 presented the report to farmer on plant production by using traditional farming system. Let the DMs Ω_1 and Ω_2 select two sets of attribute $A = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ and $B = \{\mu_1, \mu_2, , \mu_3\}$, respectively. Then DMs construct two SMSs named as Ω_A and Ω_B over H given by

$$\Omega_A = \{(\mu_1, [\frac{30}{\omega_1}, \frac{25}{\omega_2}, \frac{30}{\omega_4}]), (\mu_2, [\frac{25}{\omega_2}, \frac{28}{\omega_3}, \frac{30}{\omega_4}]), (\mu_3, [\frac{30}{\omega_4}]), (\mu_4, H)\} \text{ and}$$

$$\Omega_B = \{(\mu_1, [\frac{30}{\omega_1}, \frac{25}{\omega_2}]), (\mu_2, [\frac{25}{\omega_2}, \frac{28}{\omega_3}]), (\mu_3, [\frac{30}{\omega_4}])\}.$$

The first SMS Ω_A can be written as:

Ω_A	μ_1	μ_2	μ_3	μ_4
ω_1	30	0	0	30
ω_2	25	25	0	25
ω_3	0	28	0	28
ω_4	30	30	30	30

The second SMS Ω_B can be written as:

Ω_B	μ_1	μ_2	μ_3
ω_1	30	0	0
ω_2	25	25	0
ω_3	0	28	0
ω_4	0	0	30

Here we make a SMS-topology on Ω_A as $\widehat{\tau} = \{\Omega_\phi, \Omega_A, \Omega_B\}$, where Ω_ϕ is an empty SMS. Now we find the aggregate fuzzy soft sets Γ_A and Γ_B given by

$\Gamma_A = \{(\mu_1, \{\frac{0.35}{\omega_1}, \frac{0.29}{\omega_2}, \frac{0.35}{\omega_4}\}), (\mu_2, \{\frac{0.30}{\omega_2}, \frac{0.33}{\omega_3}, \frac{0.36}{\omega_4}\}), (\mu_3, \{\frac{1}{\omega_4}\}), (\mu_4, \{\frac{0.26}{\omega_1}, \frac{0.22}{\omega_2}, \frac{0.24}{\omega_3}, \frac{0.26}{\omega_4}\})\}$ and

$\Gamma_B = \{(\mu_1, \{\frac{0.54}{\omega_1}, \frac{0.45}{\omega_2}\}), (\mu_2, \{\frac{0.47}{\omega_2}, \frac{0.52}{\omega_3}\}), (\mu_3, \{\frac{1}{\omega_4}\})\}$.

The fuzzy soft set Γ_A can be written as:

Γ_A	μ_1	μ_2	μ_3	μ_4
ω_1	0.35	0	0	0.26
ω_2	0.29	0.30	0	0.22
ω_3	0	0.33	0	0.24
ω_4	0.35	0.36	1	0.26

The fuzzy soft set Γ_B can be written as:

Γ_B	μ_1	μ_2	μ_3
ω_1	0.54	0	0
ω_2	0.45	0.47	0
ω_3	0	0.52	0
ω_4	0	0	1

We apply here 'OR' operation on Γ_A and Γ_B, then we get $4 * 3 = 12$ attributes of the form $\mu_{ij} = (\mu_i, \mu_j),\ \forall\ i = 1, 2, 3, 4$ and $j = 1, 2, 3$. We find the fuzzy soft set for the set of attributes $A \times B = \{\mu_{11}, \mu_{12}, \mu_{13}, \mu_{21}, \mu_{22}, \mu_{23}, \mu_{31}, \mu_{32}, \mu_{33}, \mu_{41}, \mu_{42}, \mu_{43}\}$. After applying 'OR' operation we get fuzzy soft set $\Gamma_A \vee \Gamma_B$ given as:

$\Gamma_A \vee \Gamma_B = \{(\mu_{11}, \{\frac{0.54}{\omega_1}, \frac{0.45}{\omega_2}, \frac{0}{\omega_3}, \frac{0.35}{\omega_4}\}), (\mu_{12}, \{\frac{0.35}{\omega_1}, \frac{0.47}{\omega_2}, \frac{0.52}{\omega_3}, \frac{0.35}{\omega_4}\}), (\mu_{13}, \{\frac{0.35}{\omega_1}, \frac{0.29}{\omega_2}, \frac{0}{\omega_3}, \frac{1}{\omega_4}\}),$
$(\mu_{21}, \{\frac{0.54}{\omega_1}, \frac{0.45}{\omega_2}, \frac{0.33}{\omega_3}, \frac{0.36}{\omega_4}\}), (\mu_{22}, \{\frac{0}{\omega_1}, \frac{0.47}{\omega_2}, \frac{0.52}{\omega_3}, \frac{0.36}{\omega_4}\}), (\mu_{23}, \{\frac{0}{\omega_1}, \frac{0.30}{\omega_2}, \frac{0.33}{\omega_3}, \frac{1}{\omega_4}\}),$
$(\mu_{31}, \{\frac{0.54}{\omega_1}, \frac{0.45}{\omega_2}, \frac{0}{\omega_3}, \frac{1}{\omega_4}\}), (\mu_{32}, \{\frac{0}{\omega_1}, \frac{0.47}{\omega_2}, \frac{0.52}{\omega_3}, \frac{1}{\omega_4}\}), (\mu_{33}, \{\frac{0}{\omega_1}, \frac{0}{\omega_2}, \frac{0}{\omega_3}, \frac{1}{\omega_4}\}),$
$(\mu_{41}, \{\frac{0.54}{\omega_1}, \frac{0.45}{\omega_2}, \frac{0.24}{\omega_3}, \frac{0.26}{\omega_4}\}), (\mu_{42}, \{\frac{0.26}{\omega_1}, \frac{0.47}{\omega_2}, \frac{0.52}{\omega_3}, \frac{0.26}{\omega_4}\}), (\mu_{43}, \{\frac{0.26}{\omega_1}, \frac{0.22}{\omega_2}, \frac{0.24}{\omega_3}, \frac{1}{\omega_4}\})\}.$

Now the tabular form of $\Gamma_A \vee \Gamma_B$ is written as:

$\Gamma_A \vee \Gamma_B$	μ_{11}	μ_{12}	μ_{13}	μ_{21}	μ_{22}	μ_{23}	μ_{31}	μ_{32}	μ_{33}	μ_{41}	μ_{42}	μ_{43}
ω_1	0.54	0.35	0.35	0.54	0	0	0.54	0	0	0.54	0.26	0.26
ω_2	0.45	0.47	0.29	0.45	0.47	0.30	0.45	0.47	0	0.45	0.47	0.22
ω_3	0	0.52	0	0.33	0.52	0.33	0	0.52	0	0.24	0.52	0.24
ω_4	0.35	0.35	1	0.36	0.36	1	1	1	1	0.26	0.26	1

Now we find the comparison-table of fuzzy soft set $\Gamma_A \vee \Gamma_B$ by using the algorithm which is given by Roy and Maji in (2007). The comparison-table is given below.

	ω_1	ω_2	ω_3	ω_4
ω_1	12	6	6	5
ω_2	6	12	6	6
ω_3	6	7	12	3
ω_4	9	6	9	12

Here we calculate the column-sum (t_i) and row-sum (r_i) after that we calculate the score (R_i) for each $\omega_i,\ i = 1, 2, 3, 4$.

	row-sum (r_i)	column-sum (t_i)	score $(R_i = r_i - t_i)$
ω_1	29	33	-4
ω_2	30	31	-1
ω_3	28	33	-5
ω_4	36	26	10

Table 2: Tabular form of score score $(R_i = r_i - t_i)$

From Table 2, we see that the topmost score is 10 which is gained by ω_4. Which shows that neem plant is selected to safe the fields from pests. Now problem is that where to grow the neem plants to protect the field from pests. To solve this problem, we find the SMS boundary of soft open multi-sets.

If the SMS boundary of at least one soft open multi-sets is not a null SMSs and contains $\frac{30}{\omega_4}$ in non-null μ-approximate elements, $\forall\ \mu \in S$, then neem plants can grow on the corners of the field. If the SMS boundary of all soft open multi-sets are null SMSs, then neem plants cannot grow on the corners of the field.

Now compute the SMS boundary of Ω_ϕ, Ω_A and Ω_B given as:

$$\Omega_\phi^{\widehat{b}} = \Omega_\phi,\ \Omega_A^{\widehat{b}} = \Omega_\phi \text{ and } \Omega_B^{\widehat{b}} = \overline{\Omega_B} \widehat{\cap} \overline{\Omega_B^c} = \Omega_A \widehat{\cap} \Omega_B^c = \Omega_B^c = \{(\mu_1, [\tfrac{30}{\omega_4}]), (\mu_2, [\tfrac{30}{\omega_4}]), (\mu_4, H)\}.$$

Which shows that $\Omega_B^{\widehat{b}}$ contains $\frac{30}{\omega_4}$ in non-null μ-approximate elements $\forall\ \mu \in S$. So farmer decides to grow neem plants on the corners of field.
The attention should be given to grow neem plants as a reassuring choice to exchange agrochemicals in agriculture pest control. Neem can conduce to acceptable development and the determination of pest control problems in agriculture which can be best alternative to plant fertilizer.

The proposed Algorithm 1 is used in the environment of SMSs information for the selection of best alternative of biopesticides and the results are compared as indicated in the Table 3.

Method	Ranking of alternatives	The optimal alternative
Algorithm 1 (Proposed)	$\omega_4 \succ \omega_2 \succ \omega_1 \succ \omega_3$	ω_4
Algorithm (Çağman *et al.*, 2011)	$\omega_4 \succ \omega_2 \succ \omega_1 \succ \omega_3$	ω_4
Algorithm (Riaz *et al.*, 2019)	$\omega_4 \succ \omega_2 \succ \omega_1 \succ \omega_3$	ω_4

Table 3: Comparison of final ranking with existing methods using Algorithm 1.

4.2 MCDM by using SMS-topology for the selection of best textile company

We present two modified algorithms based on SMS-topology for a decision-making problem. At the end, we show the comparison of ranking of objects obtained by Algorithm 2 and Algorithm 3. Furthermore we present another interesting application in agriculture for decision-making to find the optimal choice by using SMS-topology and boundaries of soft open multi-set.

Algorithm 2 The selection of best textile company

Input:
Step 1: Consider a universe of multi-set (MS) U.
Step 2: A set E of attributes.
Step 3: Construct SMS F_A and F_B.
Output:
Step 4: Write SMS-topology $\widetilde{\tau}$ in which F_A and F_B are open SMSs in $\widetilde{\tau}$.
Step 5: Write the aggregate multi-sets of all open SMSs by using the formula, $F_A^* = [\frac{F_A^*(\Omega_i)}{\Omega_i} : \Omega_i \in X]$, where $F_A^*(\Omega_i) = \Sigma_j \Omega_{ij}$.
Step 6: Add F_A^* and F_B^* to find decision MS.
Step 7: Select the object with greatest multiplicity determined by $\max F_{A\oplus B}^*(\sigma)$.

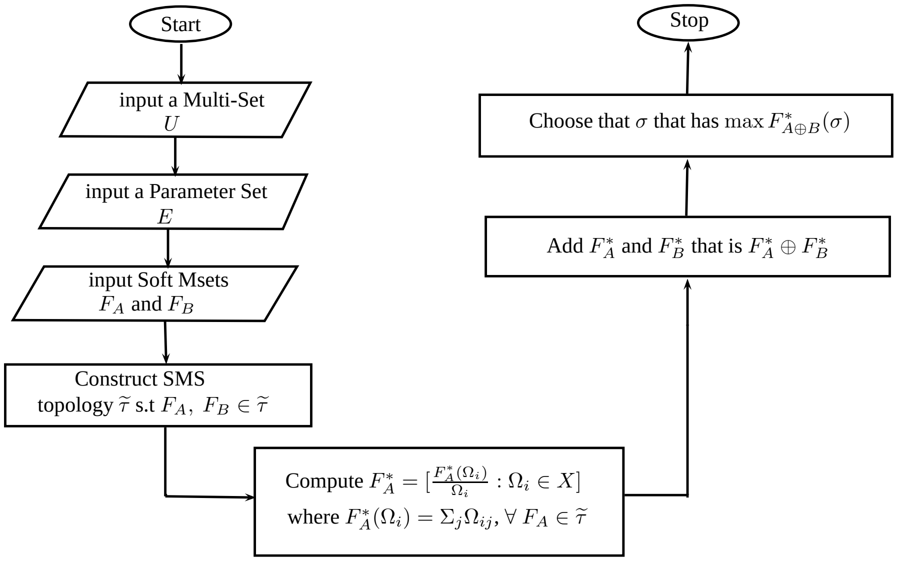

Figure 2: Graphical representation of Algorithm 2

Graphical representation of Algorithm 2 is shown in the Figure 2. Here we introduce another algorithm for SMS-topology in decision-making.
Now we give Algorithm 3 and compare the optimal decision obtained by Algorithm 2.

Algorithm 3 The award of performance

Input:
Step 1: Consider a universe of multi-set U.
Step 2: A set E of attributes.
Step 3: Construct SMSs F_A and F_B.
Output:
Step 4: Write SMS-topology $\widetilde{\tau}$ containing F_A and F_B as open SMSs in $\widetilde{\tau}$.
Step 5: Find the cardinal MSs of all open SMSs by using the formula,
$cF_A = [\frac{cF_A(\lambda_i)}{\lambda_i} : \lambda_i \in E]$, where $cF_A(\lambda_i) = \Sigma_i \Omega_i j$.
Step 6: Find the aggregate multi-sets by using the formula,
$\dddot{M}_{F_A^*} = \dddot{M}_{F_A} * M_{cF_A}^t, \qquad \rightarrow (1)$
where $\dddot{M}_{F_A},\ \dddot{M}_{cF_A}$ and $\dddot{M}_{F_A^*}$ are representation matrices of $F_A,\ cF_A$ and F_A^*, respectively.
Step 7: Adding F_A^* and F_B^* to find decision mset.
Step 8: Select the object that has greatest multiplicity i.e. $\max F_{A\oplus B}^*(\sigma)$.

A brief sketch of Algorithm 3 is given in the Figure 3.

Assume that government of a country is interested to give the "award of performance" to best textile company of country to appreciate the contribution of the company. Let $U = [\frac{2}{\Omega_1}, \frac{2}{\Omega_2}, \frac{1}{\Omega_3}, \frac{1}{\Omega_4}, \frac{1}{\Omega_5}, \frac{1}{\Omega_6}, \frac{1}{\Omega_7}]$ be the multi-set of big textile companies of the state, and the multiplicity of $\Omega_i,\ i = 1, 2, ..., 7$ denotes the

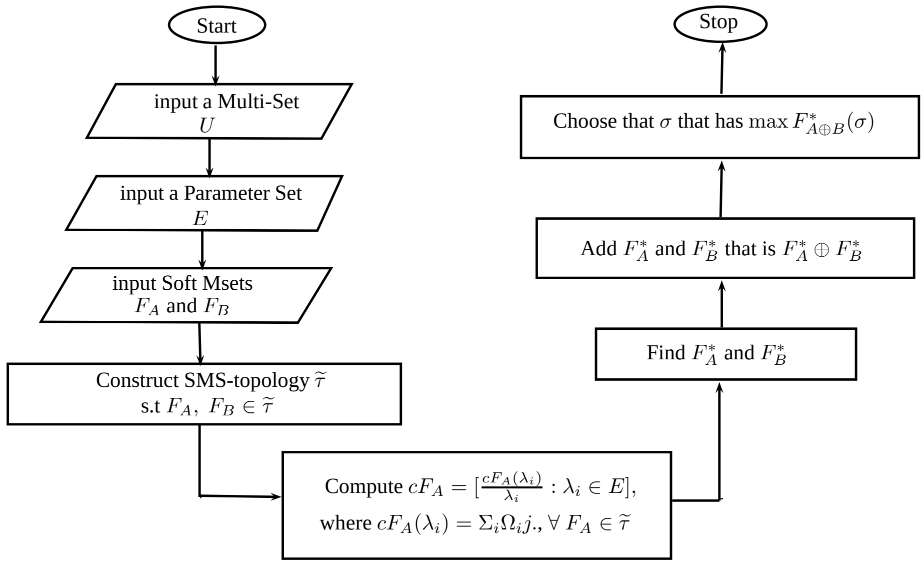

Figure 3: Graphical representation of Algorithm 3

number of branches of company Ω_i that are selected for the award. Let $X = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7\}$ be the support set of U.
The set of parameters is given as $E = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ where
$\lambda_1 =$ best hosiery,
$\lambda_2 =$ best export,
$\lambda_3 =$ healthy working environment,
$\lambda_4 =$ use of modern technology,
$\lambda_5 =$ expert workers.
We here use the following Algorithm 2 to select the best company of the state for the "award of performance.

The DMs Ω_1 and Ω_2 construct two squads named as squad-Ω_1 and squad-Ω_2, respectively. Then they choose two sets of attributes $A = \{\lambda_1, \lambda_2, \lambda_3\}$ and $B = \{\lambda_1, \lambda_2\}$ and use them to construct soft multi-sets (SMSs) F_A and F_B over U given by
$F_A = \{(\lambda_1, [\frac{2}{\Omega_1}, \frac{2}{\Omega_2}, \frac{1}{\Omega_3}, \frac{1}{\Omega_4}]), (\lambda_2, [\frac{2}{\Omega_1}, \frac{2}{\Omega_2}, \frac{1}{\Omega_6}, \frac{1}{\Omega_7}]), (\lambda_3, [\frac{2}{\Omega_1}, \frac{2}{\Omega_2}, \frac{1}{\Omega_5}, \frac{1}{\Omega_6}, \frac{1}{\Omega_7}])\}$ and
$F_B = \{(\lambda_1, [\frac{2}{\Omega_1}, \frac{2}{\Omega_2}, \frac{1}{\Omega_4}]), (\lambda_2, [\frac{2}{\Omega_2}, \frac{1}{\Omega_6}, \frac{1}{\Omega_7}])\}$.
The 1st SMS F_A can be written as

F_A	λ_1	λ_2	λ_3
Ω_1	2	2	2
Ω_2	2	2	2
Ω_3	1	0	0
Ω_4	1	0	0
Ω_5	0	0	1
Ω_6	0	1	1
Ω_7	0	1	1

The 2nd SMS F_B can be written as

F_B	λ_1	λ_2
Ω_1	2	0
Ω_2	2	2
Ω_3	0	0
Ω_4	1	0
Ω_5	0	0
Ω_6	0	1
Ω_7	0	1

Now we construct a SMS-topology as

$$\widetilde{\tau} = \{F_\phi, F_A, F_B, F_{\widetilde{E}}\},$$

where F_ϕ and $F_{\widetilde{E}}$ are empty soft and absolute soft msets, respectively.
Write aggregate multi-sets of all open SMSs given by
$F_A^* = [\frac{6}{\Omega_1}, \frac{6}{\Omega_2}, \frac{1}{\Omega_3}, \frac{1}{\Omega_4}, \frac{1}{\Omega_5}, \frac{2}{\Omega_6}, \frac{2}{\Omega_7}]$,
$F_B^* = [\frac{2}{\Omega_1}, \frac{4}{\Omega_2}, \frac{1}{\Omega_4}, \frac{1}{\Omega_6}, \frac{1}{\Omega_7}]$,
$F_\phi^* = [\frac{0}{\Omega_1}, \frac{0}{\Omega_2}, \frac{0}{\Omega_3}, \frac{0}{\Omega_4}, \frac{0}{\Omega_5}, \frac{0}{\Omega_6}, \frac{0}{\Omega_7}]$
and $F_{\widetilde{E}}^* = [\frac{10}{\Omega_1}, \frac{10}{\Omega_2}, \frac{5}{\Omega_3}, \frac{5}{\Omega_4}, \frac{5}{\Omega_5}, \frac{5}{\Omega_6}, \frac{5}{\Omega_7}]$.
In order to evaluate decision multi-set, The DMs added the sets F_A^* and F_B^*.
Thus $F_{A\oplus B}^*(\sigma) = F_A^*(\sigma) + F_B^*(\sigma), \quad \forall\, \sigma \in X$.
Thus $F_A^* \oplus F_B^* = [\frac{8}{\Omega_1}, \frac{10}{\Omega_2}, \frac{1}{\Omega_3}, \frac{2}{\Omega_4}, \frac{1}{\Omega_5}, \frac{3}{\Omega_6}, \frac{3}{\Omega_7}]$.

Since $\max F_{A\oplus B}^*(\sigma) = 10$ which shows that Ω_2 has the highest multiplicity, so Ω_2 is chosen for the "award of performance".

Next we use Algorithm 3 on the same data as above and then compare the optimal results.
The DMs Ω_1 and Ω_2 consider SMSs (data same as above) F_A and F_B over U given by
$F_A = \{(\lambda_1, [\frac{2}{\Omega_1}, \frac{2}{\Omega_2}, \frac{1}{\Omega_3}, \frac{1}{\Omega_4}]), (\lambda_2, [\frac{2}{\Omega_1}, \frac{2}{\Omega_2}, \frac{1}{\Omega_6}, \frac{1}{\Omega_7}]), (\lambda_3, [\frac{2}{\Omega_1}, \frac{2}{\Omega_2}, \frac{1}{\Omega_5}, \frac{1}{\Omega_6}, \frac{1}{\Omega_7}])\}$
and $F_B = \{(\lambda_1, [\frac{2}{\Omega_1}, \frac{2}{\Omega_2}, \frac{1}{\Omega_4}]), (\lambda_2, [\frac{2}{\Omega_2}, \frac{1}{\Omega_6}, \frac{1}{\Omega_7}])\}$.
Again consider first SMS F_A given as

F_A	λ_1	λ_2	λ_3
Ω_1	2	2	2
Ω_2	2	2	2
Ω_3	1	0	0
Ω_4	1	0	0
Ω_5	0	0	1
Ω_6	0	1	1
Ω_7	0	1	1

Now consider second SMS F_B given as

F_B	λ_1	λ_2
Ω_1	2	0
Ω_2	2	2
Ω_3	0	0
Ω_4	1	0
Ω_5	0	0
Ω_6	0	1
Ω_7	0	1

Now we make a SMS-topology as

$$\widetilde{\tau} = \{F_\phi, F_A, F_B, F_{\widetilde{E}}\},$$

where F_ϕ and $F_{\widetilde{E}}$ are empty soft and absolute soft msets, respectively.
Here we find the cardinal msets of all soft open msets given by
$cF_A = [\frac{6}{\lambda_1}, \frac{6}{\lambda_2}, \frac{7}{\lambda_3}]$,
$cF_B = [\frac{5}{\lambda_1}, \frac{4}{\lambda_2}]$,
$cF_\phi = [\frac{0}{\lambda_1}, \frac{0}{\lambda_2}, \frac{0}{\lambda_3}, \frac{0}{\lambda_4}, \frac{0}{\lambda_5}]$
and $cF_{\widetilde{E}} = [\frac{9}{\lambda_1}, \frac{9}{\lambda_2}, \frac{9}{\lambda_3}, \frac{9}{\lambda_4}, \frac{9}{\lambda_5}]$.
The aggregated multi-set F_A^* is calculated by first decision maker by using (1),

$$\dddot{M}_{F_A^*} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 38 \\ 38 \\ 6 \\ 6 \\ 7 \\ 13 \\ 13 \end{bmatrix}$$

that means, $F_A^* = [\frac{38}{\Omega_1}, \frac{38}{\Omega_2}, \frac{6}{\Omega_3}, \frac{6}{\Omega_4}, \frac{7}{\Omega_5}, \frac{13}{\Omega_6}, \frac{13}{\Omega_7}]$.
Furthermore, the aggregate multi-set for F_B is calculated by second decision maker,

$$\ddot{M}_{F_B^*} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 0 \\ 5 \\ 0 \\ 4 \\ 4 \end{bmatrix}$$

which is, $F_B^* = [\frac{10}{\Omega_1}, \frac{18}{\Omega_2}, \frac{0}{\Omega_3}, \frac{5}{\Omega_4}, \frac{0}{\Omega_5}, \frac{4}{\Omega_6}, \frac{4}{\Omega_7}]$.
Now we find the final decision multi-set by adding F_A^* and F_B^* only.
Thus $F_{A\oplus B}^*(\sigma) = F_A^*(\sigma) + F_B^*(\sigma), \quad \forall\, \sigma \in X$.
Thus $F_A^* \oplus F_B^* = [\frac{48}{\Omega_1}, \frac{56}{\Omega_2}, \frac{6}{\Omega_3}, \frac{11}{\Omega_4}, \frac{7}{\Omega_5}, \frac{17}{\Omega_6}, \frac{17}{\Omega_7}]$.
Since $\max F_{A\oplus B}^*(\sigma) = 56$ which shows that Ω_2 has the greatest multiplicity, so Ω_2 is chosen for the "award of performance". It is interesting to note that Algorithm 2 and Algorithm 3 provides the same optimal decision.

The proposed Algorithm 2 and Algorithm 3 are used in the environment of soft multi-sets information systems for the award of performance and the results are compared with existing methods as indicated in the Table 4.

Method	Ranking of alternatives	The optimal alternative
Algorithm 2 (Proposed)	$\Omega_2 \succ \Omega_1 \succ \Omega_6 = \Omega_7 \succ \Omega_4 \succ \Omega_5 \succ \Omega_3$	Ω_2
Algorithm 3 (Proposed)	$\Omega_2 \succ \Omega_1 \succ \Omega_6 = \Omega_7 \succ \Omega_4 \succ \Omega_5 \succ \Omega_3$	Ω_2
Algorithm (Çağman *et al.*, 2011)	$\Omega_2 \succ \Omega_1 \succ \Omega_6 \succ \Omega_7 \succ \Omega_4 \succ \Omega_5 \succ \Omega_3$	Ω_2
Algorithm (Riaz *et al.*, 2011)	$\Omega_2 \succ \Omega_1 \succ \Omega_6 \succ \Omega_7 \succ \Omega_4 \succ \Omega_5 \succ \Omega_3$	Ω_2

Table 4: Comparison of final ranking by using Algorithm 2 and Algorithm 3

The comparison analysis of final ranking determined by Algorithm 2, Algorithm 3, Çağman *et al.* (2011) and Riaz *et al.* (2011) is also shown by multiple bar chart in the Figure 4.

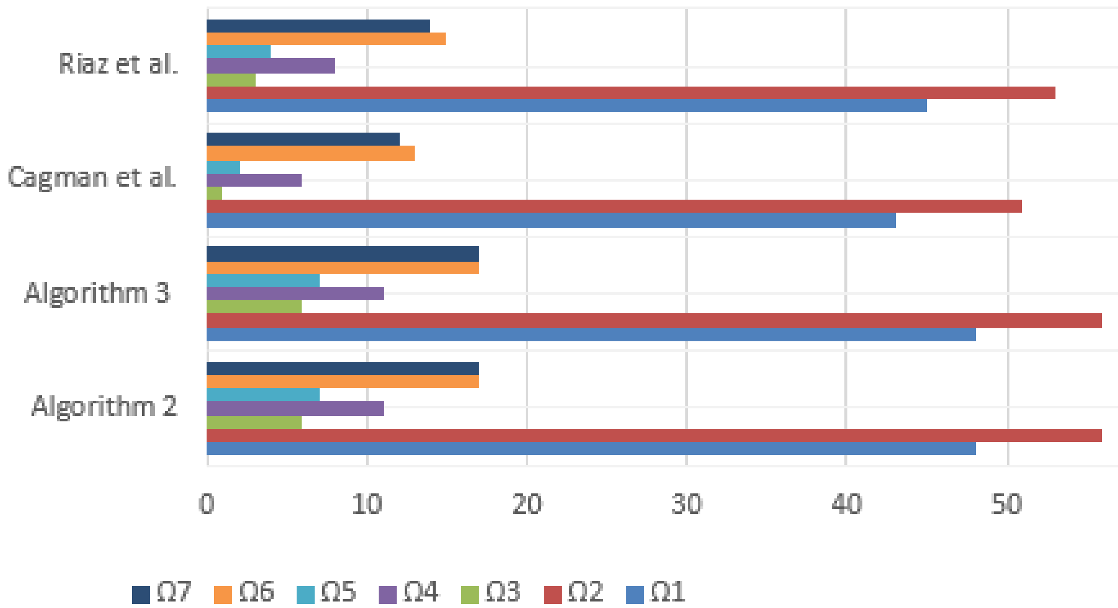

Figure 4: Multiple bar chart view of final ranking

5 Conclusion

The algebraic and topological structures of soft multi-sets (SMSs) are quite different from traditional crisp sets. Moreover the MCDM methods developed under rough sets, fuzzy sets and soft sets do not deal with real life situations under the universe of soft multi-sets. Due to the repetition of objects in the universe of soft multi-sets there is a need to develop novel MCDM methods. The goal of this article is deal with these challenges and to extend the notion of SMS-topology towards MCDM problems. We initiated the idea of SMS-topology which is defined on soft multi-sets for a fixed set of attributes. We used the idea of power whole multi-subsets of a soft multi-set in the construction of SMS-topology. The notions of SMS-basis, SMS-subspace, SMS-interior, soft multi-set closure and boundary of soft multi-set are introduced. Additionally, the concept of SMS-topology is extended to develop novel multi-criteria decision-making (MCDM) methods. To meet these objectives, Algorithm 1, Algorithm 2 and Algorithm 3 are presented for the selection of best alternative for biopesticides, for the selection of best textile company and for the award of performance, respectively. The aggregation operators are used to compute aggregate fuzzy soft sets and aggregate multi-sets. Based on proposed MCDM methods some real life applications are justified by illustrative examples. Soft multi-sets and SMS-topology have large number of applications in soft computing, decision-making, data analysis, data mining, expert systems, information aggregation and information measures.

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