SCIENTIFIC OASIS

Decision Making: Applications in Management and Engineering

Journal homepage: www.dmame-journal.org
ISSN: 2560-6018, eISSN: 2620-0104

Optimizing University Teaching Methods Using Algebraic Techniques in a Complex Fermatean Environment

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ARTICLE INFO

Article history:
Received 10 June 2024
Received in revised form 20 October 2024
Accepted 10 December 2024
Available online 30 December 2024

Keywords:
Classroom Management; CIVFFS, Algebraic Techniques; Decision-Making Process

ABSTRACT

In a university classroom, students exhibit varying levels of engagement, posing challenges for professors in delivering instruction effectively. While some students are highly engaged, others may struggle to keep pace, creating a demanding environment for the professor to manage while ensuring comprehensive learning. Consequently, the professor faces two primary options: either disregarding students with differing engagement levels or adopting an alternative teaching approach. An effective professor adapts their teaching methodology according to student engagement; however, determining the most suitable approach for the majority of students within a class is a complex task. The selection of an appropriate teaching mode is further complicated by various uncertain factors. This study seeks to develop a structured approach for selecting classroom teaching modes based on diverse levels of student engagement while considering multiple influencing factors. To achieve this, a novel model is introduced, namely the complex interval-valued Fermatean fuzzy set (CIVFFS), which integrates the characteristics of the complex Fermatean fuzzy set (CFFS) and the interval-valued Fermatean fuzzy set (IVFFS). The CIVFFS plays a crucial role in addressing the uncertain and incomplete information associated with various factors affecting classroom teaching methods. Four methods are proposed based on algebraic t-norms and t-conorms, namely the complex interval-valued Fermatean fuzzy weighted averaging (CIVFFWA) operator, the complex interval-valued Fermatean fuzzy ordered weighted averaging (CIVFFOWA) operator, the complex interval-valued Fermatean fuzzy weighted geometric (CIVFFWG) operator, and the complex interval-valued Fermatean fuzzy ordered weighted geometric (CIVFFOWG) operator, along with their respective properties. The proposed approach enables an evaluation of its effectiveness across different scenarios. An illustrative example is provided to demonstrate the practicality and reliability of the model and methods, highlighting their applicability in real-world contexts.

1. Introduction

Educational institutions face significant challenges in keeping pace with the rapid advancement of technology, which directly influences students' learning processes. The selection of appropriate

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https://doi.org/10.31181/dmame7220241345

teaching methodologies is crucial for enhancing student learning outcomes. Interactive approaches, such as group discussions, collaborative study, and problem-solving, often yield superior results, as highly engaged students actively participate in seminars, presentations, and other interactive activities. These methods contribute to a more effective and engaging learning experience. However, students with lower engagement levels often require traditional teaching methodologies supplemented with multimedia tools or other interactive techniques to capture their attention. Selecting an appropriate teaching approach is a complex task for educators due to the diverse engagement levels among students. The choice of teaching mode is influenced by multiple factors, each characterised by uncertainty and ambiguity, making their modelling particularly challenging.

Various methods and frameworks have been developed to address uncertainty. The concept of fuzzy sets (FS) was first introduced to represent an object's degree of belonging to a set using a membership grade (MG) [1]. FS has played a significant role in reducing uncertainty in information processing. Unlike classical set theory, which operates on binary true-false logic, FS provides a more nuanced representation of real-world scenarios by assigning values between 0 and 1 to indicate varying levels of engagement. This flexibility enhances its applicability in dealing with ambiguous situations. However, FS lacks the capability to account for hesitation in complex environments. For instance, if the MG of an element is 0.4, the corresponding non-membership grade (NMG) is conventionally determined as 1 - 0.4 = 0.6. This approach does not acknowledge the independent existence of NMG, thereby restricting FS's ability to comprehensively model uncertainty in certain contexts. To overcome these limitations, [2] introduced the intuitionistic fuzzy set (IFS), with $0 \prec m + n \leq 1$, where m is called MG and n is NMG.

Later, researchers such as [3], [4], [5;6], and [7] introduced various methods, including algebraic, Einstein, and Hamacher techniques. These approaches proved valuable in enhancing decision-making processes, demonstrating their effectiveness in handling complex decisions. Building on the concept of IFS, [8] introduced the interval-valued intuitionistic fuzzy set (IVIFS), further expanding its applicability. This extension allows for more flexible handling of uncertainty in various situations. In IVIFS, element is presented as: $([m, n], [d, g])$ under restriction $0 \leq n + g \leq 1$. The researchers in [9] presented a new concept called Pythagorean fuzzy set (PyFS), under condition: $0 \prec m^2 + n^2 \leq 1$. The authors in [10;11] proposed various techniques utilising PyFNs and applied them to the decision-making process. Building on this, [12] introduced the interval-valued Pythagorean fuzzy set (IVPyFS), providing a more flexible framework for representing and managing uncertainty. In IVPyFS, each element is expressed as: $([m, n], [d, g])$ with $0 \leq n^2 + g^2 \leq 1$. The authors in [13;14] introduced new methods based on IVPyFNs, successfully applying them to decision-making processes. Their findings demonstrated how IVPyFNs enhance both the accuracy and flexibility of decision-making. Advancing this field, [15] proposed the Fermatean fuzzy set (FFS), an extension of PyFS that relaxes its inherent limitations. Building on FFNs, [16] introduced Dombi techniques, effectively applying them to various decision-making scenarios. Further extending this framework, [17] developed the interval-valued Fermatean fuzzy set (IVFFS), which enhances the adaptability of IVPyFS by modifying the constraint from $0 \leq n^2 + g^2 \leq 1$ to $0 \leq n^3 + g^3 \leq 1$. This advancement significantly improves the representation of uncertainty and fuzziness, leading to more precise and flexible solutions for complex decision-making problems.

Existing models, such as FS, IFS, PyFS, FFS, IVIFS, IVPyFS, and IVFFS, primarily focus on decision-making problems and effectively handle uncertainty and vagueness. However, they struggle to capture partial ignorance and adapt to changes over time, particularly in complex datasets such as biometrics, medical research, and multimedia. This highlights the need for more advanced models that can manage incomplete, dynamic, and imprecise information more effectively. To address these limitations, the complex fuzzy set (CFS) was introduced to improve handling of partial ignorance and track variations in dynamic datasets, particularly in areas such as audio processing, healthcare, and image analysis [18]. Further advancements led to the introduction of the complex intuitionistic fuzzy set (CIFS), which extended these capabilities [19]. In CIFS, each element is defined as: $\left(me^{ip}, ne^{iq}\right)$ with $0 \leq m + n \leq 1$ and $0 \leq \frac{p}{2\pi} + \frac{q}{2\pi} \leq 1$. Later, several researchers developed new methods using CIFNs, enhancing the handling of uncertainty and ambiguity in decision-making across various fields [20-22]. These advancements represent significant progress in managing complex information in diverse applications.

Further developments led to the introduction of the CIVIFS, an extension of the traditional IVIFS, which provides a more effective representation of uncertainty by incorporating both real and imaginary components [23]. In CIVIFS, each element is presented as: $\left([m,n]e^{i[p,q]},[d,g]e^{i[r,s]}\right)$ with $0 \leq n+g \leq 1$ and $0 \leq \frac{q}{2\pi}+\frac{s}{2\pi} \leq 1$. Several advanced techniques using CIVIFNs were developed to enhance decision-making, providing greater flexibility and precision in handling complex and uncertain information [24]. The complex Pythagorean fuzzy set (CPyFS) was later introduced as an extension of CIFS, improving its flexibility, adaptability, and ability to manage complex uncertainties [25]. In CPyFS, the condition is defined as: $0 \leq m^2+n^2 \leq 1$ and $0 \leq \left(\frac{p}{2\pi}\right)^2+\left(\frac{q}{2\pi}\right)^2 \leq 1$. A series of new approaches focusing on CPyFNs have been developed and applied in decision-making processes, demonstrating their effectiveness and applicability across various scenarios [26-29]. Further advancements led to the introduction of complex interval-valued Pythagorean fuzzy sets (CIVPyFS), which were proposed with certain limitations [30] such as: $0 \leq n^2+g^2 \leq 1$ and $0 \leq \left(\frac{q}{2\pi}\right)^2+\left(\frac{s}{2\pi}\right)^2 \leq 1$.

The CFFS was innovatively introduced, providing an advanced approach to handling uncertainty and complexity in decision-making [31]. In CFFS, each element is presented as: $\left(me^{ip},ne^{iq}\right)$ with $0 \leq m^3+n^3 \leq 1$ and $0 \leq \left(\frac{p}{2\pi}\right)^3+\left(\frac{q}{2\pi}\right)^3 \leq 1$. CFFS surpasses FFS, CIFS, and CPyFS in adaptability, reliability, and flexibility, providing enhanced performance and greater versatility. Its advanced features make it a more efficient and practical option for various applications. Building on the strengths of previous models and their aggregation operators, this study introduces CIVFFS as a more powerful tool. A set of new operators based on CIVFF-information is proposed, including CIVFFWA, CIVFFOWA, CIVFFWG, and CIVFFOWG. These operators enhance the capabilities of existing models, improving overall performance. To validate the effectiveness of these methods, an example is provided to demonstrate their application in selecting university-level teaching methods, illustrating their practical significance.

The structure of this research is organised to ensure a clear and systematic exploration of the study. Section 2 establishes the foundational definitions necessary for understanding the key concepts. Section 3 introduces CIVFFS along with their primary operational laws. Section 4 presents newly developed operators—CIVFFWA, CIVFFOWA, CIVFFWG, and CIVFFOWG—detailing their functionalities and applications. Section 5 focuses on the practical applications of these operators, demonstrating their real-world significance. Section 6 provides an illustrative example to highlight the utility of these techniques in a concrete scenario. Section 7 offers a comparative analysis of the proposed methods, while Section 8 concludes with a summary of key findings and contributions.

2. Preliminaries

This section introduces fundamental definitions that will be utilised throughout the research.

Definition 1 [31]:The CFFS $\mathbb{C}$ on a universal set T is the set of ordered pairs having the mathematical form as: $\mathbb{C}=\left\{\left\langle t,m_{\mathbb{C}}(t)e^{ip_{\mathbb{C}}(t)},n_{\mathbb{C}}(t)e^{iq_{\mathbb{C}}(t)}\right\rangle \middle| t\in T\right\}$, where $m_{\mathbb{C}}(t):T\to[0,1]$ and $n_{\mathbb{C}}(t):T\to[0,1]$ present the grade of complex valued membership and complex valued non-membership of the element t in the set $\mathbb{C}$ with $0 \leq (m)^3+(n)^3 \leq 1$ and $0 \leq \left(\frac{p}{2\pi}\right)^3+\left(\frac{q}{2\pi}\right)^3 \leq 1$ with $p_{\mathbb{C}}(t)\in[0,2\pi]$, $q_{\mathbb{C}}(t)\in[0,2\pi]$ respectively, Furthermore $i=\sqrt{-1}$ be a unit circle and let$\pi_{\mathbb{C}}(t)=\sqrt[3]{1-\left(\left(m_{\mathbb{C}}(t)\right)^3+\left(n_{\mathbb{C}}(t)\right)^3\right)}e^{\sqrt[3]{1-\left(\left(p_{\mathbb{C}}(t)\right)^3+\left(q_{\mathbb{C}}(t)\right)^3\right)}}$, then the term $\pi_{\mathbb{C}}(t)$ is called the grade of indeterminacy or hesitancy of the element t to T, $\forall t \in T$.

Definition 2 [31]: Let $Њ_j = \left(m_j e^{ip_j}, n_j e^{iq_j}\right)(j = 1,2)$ be a family of two CFFNs and a real number $\chi \succ 0$, then the following theoretical operational laws hold:

i) $$Њ_1 \oplus Њ_2 = \left(\sqrt[3]{m_1^3 + m_2^3 - m_1^3 m_2^3}\, e^{i\left(\sqrt[3]{\left(\frac{p_1}{2\pi}\right)^3 + \left(\frac{p_2}{2\pi}\right)^3 - \left(\frac{p_1}{2\pi}\right)^3\left(\frac{p_2}{2\pi}\right)^3}\right)}, (n_1 n_2) e^{i\left(\frac{q_1}{2\pi}\right)\left(\frac{q_2}{2\pi}\right)} \right)$$

ii) $$Њ_1 \otimes Њ_2 = \left((m_1 m_2) e^{i\left(\frac{p_1}{2\pi}\right)\left(\frac{p_2}{2\pi}\right)}, \sqrt[3]{n_1^3 + n_2^3 - n_1^3 n_2^3}\, e^{i\left(\sqrt[3]{\left(\frac{q_1}{2\pi}\right)^3 + \left(\frac{q_2}{2\pi}\right)^3 - \left(\frac{q_1}{2\pi}\right)^3\left(\frac{q_2}{2\pi}\right)^3}\right)} \right)$$

iii) $$\chi(Њ) = \left(\sqrt[3]{1 - (1 - m^3)^{\chi}}\, e^{i\left(\sqrt[3]{1 - \left(1 - \left(\frac{p}{2\pi}\right)^3\right)^{\chi}}\right)}, n^{\chi} e^{i\left(\frac{q}{2\pi}\right)^{\chi}} \right)$$

iv) $$(Њ)^{\chi} = \left((m)^{\chi} e^{i\left(\frac{p}{2\pi}\right)^{\chi}}, \sqrt[3]{1 - (1 - n^3)^{\chi}}\, e^{i\left(\sqrt[3]{1 - \left(1 - \left(\frac{q}{2\pi}\right)^3\right)^{\chi}}\right)} \right)$$

Definition 3: If $Њ = \left(m e^{ip}, n e^{iq}\right)$ be a CFFN, then the score value of the CFFN is defined as: $score(Њ) = (m^3 - n^3) + \frac{1}{8\pi^3}(p^3 - q^3)$ with limitation: $score(Њ) \in [-2, 2]$

3. Complex Interval-Valued Fermatean Fuzzy Set

This section explores the concept of CIVFFSs, including their operational laws, scoring functions, and accuracy measures. Fundamental results are developed to demonstrate the flexibility and adaptability of this approach. CIVFFSs offer an improved method for managing uncertainty and imprecision in decision-making, enhancing problem-solving in complex scenarios.

Definition 4: The CIVFFS ***F*** on a universal discourse ***X*** can be mathematically defines as: $F = \{\tau, ([V_F^-(\tau), V_F^+(\tau)], [U_F^-(\tau), U_F^+(\tau)]) | \tau \in X\}$, where $V_F^-(\tau)$, $V_F^+(\tau)$, $U_F^-(\tau)$, $U_F^+(\tau)$ represent the degrees associated with the lower and upper bounds of membership and non-membership defined as: $V_F^-(\tau) = z_1^{-ve} = m_F(\tau) e^{ip_F(\tau)}$, $V_F^+(\tau) = x_1^{+ve} = n_F(\tau) e^{iq_F(\tau)}$ with $|x_1^{-ve}| \prec |x_1^{+ve}|$, while $U_F^-(\tau) = x_2^{-ve} = d_F(\tau) e^{ir_F(\tau)}$, $U_F^+(\tau) = x_2^{+ve} = g_F(\tau) e^{is_F(\tau)}$ with $|x_2^{-ve}| \prec |x_2^{+ve}|$. All of the amplitude terms $m_F(\tau)$, $n_F(\tau)$, $d_F(\tau)$, $g_F(\tau)$ are belong to the closed interval [0,1] and satisfying the conditions: $m_F(\tau) \prec n_F(\tau)$ and $d_F(\tau) \prec g_F(\tau)$ with $0 \prec \left(n_F(\tau)\right)^3 + \left(g_F(\tau)\right)^3 \leq 1$, $\forall \quad \tau \in X$. Similarly the phase terms $p_F(\tau)$, $q_F(\tau)$, $r_F(\tau)$, $s_F(\tau)$ are real valued numbers which lie in the interval $[0,2\pi]$ and satisfying the conditions: $0 \prec \left(\frac{\lambda_F(\tau)}{2\pi}\right)^3 + \left(\frac{\ell_F(\tau)}{2\pi}\right)^3 \leq 1$ with $p_F(\tau) \prec q_F(\tau)$ and $r_F(\tau) \prec s_F(\tau)$. Thus, CIVFFS ***F*** can be represented mathematically on ***X*** as:

$$F = \left\{\tau, \left([m_F(\tau), n_F(\tau)] e^{i[p_F(\tau), q_F(\tau)]}, [d_F(\tau), g_F(\tau)] e^{i[r_F(\tau), s_F(\tau)]}\right) | \tau \in X\right\} \quad (1)$$

Moreover, if $\pi_F(\tau) = \left(\pi_F^-(\tau), \pi_F^+(\tau)\right) e^{i\left(\kappa_F^-(\tau), \kappa_F^+(\tau)\right)}$, then it is called the CIVFF-index of τ to ***F***, where $\pi_F^-(\tau) = \sqrt[3]{1 - \left(\left(n_F(\tau)\right)^3 + \left(g_F(\tau)\right)^3\right)}$, $\pi_F^+(\tau) = \sqrt[3]{1 - \left(\left(m_F(\tau)\right)^3 + \left(d_F(\tau)\right)^3\right)}$, $\kappa_F^-(\tau) = \sqrt[3]{1 - \left(\left(q_F(\tau)\right)^3 + \left(s_F(\tau)\right)^3\right)}$ and $\kappa_F^+(\tau) = \sqrt[3]{1 - \left(\left(p_F(\tau)\right)^3 + \left(r_F(\tau)\right)^3\right)}$. Furthermore, the complex interval-valued Fermatean fuzzy number can be expressed numerically as: $Ћ = \left([m,n]e^{i(p,q)}, [d,g]e^{i(r,s)}\right)$.

In fuzzy set theory, particularly in decision-making analysis, t-norms (T) and s-norms (S) are functions representing conjunction (AND) and disjunction (OR), respectively. T-norms, such as the minimum and algebraic product, model the intersection of fuzzy sets, while s-norms define their union.

Definition 5: Let ***x*** and ***y*** are any two natural numbers, and then their algebraic t-norms (T) and s-norms (S) is defined mathematically as: $T(x,y) = xy$ and $S(x,y) = x + y - xy$ respectively. Based on these algebraic norms, four operations are defined, which will be utilised in the development of aggregation operators in the subsequent section. These operations serve as the foundation for constructing effective aggregation techniques.

Definition 6: Let $Ћ_j = \left([m_j,n_j]e^{i(p_j,q_j)}, [d_j,g_j]e^{i(r_j,s_j)}\right)(j = 1,2)$ be a family of two CFFNs and a real number $\chi \succ 0$, then the following operational laws hold:

i) $$Ћ_1 \oplus Ћ_2 = \left(\begin{bmatrix} \sqrt[3]{m_1^3 + m_2^3 - m_1^3 m_2^3}, \\ \sqrt[3]{n_1^3 + n_2^3 - n_1^3 n_2^3} \end{bmatrix} e^{i\left(\begin{array}{c} \sqrt[3]{\left(\frac{p_1}{2\pi}\right)^3 + \left(\frac{p_2}{2\pi}\right)^3 - \left(\frac{p_1}{2\pi}\right)^3 \left(\frac{p_2}{2\pi}\right)^3}, \\ \sqrt[3]{\left(\frac{q_1}{2\pi}\right)^3 + \left(\frac{q_2}{2\pi}\right)^3 - \left(\frac{q_1}{2\pi}\right)^3 \left(\frac{q_2}{2\pi}\right)^3} \end{array} \right)}, \\ [d_1 d_2, g_1 g_2] e^{i\left(\left(\frac{r_1}{2\pi}\right)\left(\frac{r_2}{2\pi}\right), \left(\frac{s_1}{2\pi}\right)\left(\frac{s_2}{2\pi}\right)\right)} \right)$$

ii) $$Ћ_1 \otimes Ћ_2 = \left([m_1 m_2, n_1 n_2] e^{i\left(\left(\frac{p_1}{2\pi}\right)\left(\frac{p_2}{2\pi}\right), \left(\frac{q_1}{2\pi}\right)\left(\frac{q_2}{2\pi}\right)\right)}, \\ \begin{bmatrix} \sqrt[3]{d_1^3 + d_2^3 - d_1^3 d_2^3}, \\ \sqrt[3]{g_1^3 + g_2^3 - g_1^3 g_2^3} \end{bmatrix} e^{i\left(\begin{array}{c} \sqrt[3]{\left(\frac{r_1}{2\pi}\right)^3 + \left(\frac{r_2}{2\pi}\right)^3 - \left(\frac{r_1}{2\pi}\right)^3 \left(\frac{r_2}{2\pi}\right)^3}, \\ \sqrt[3]{\left(\frac{s_1}{2\pi}\right)^3 + \left(\frac{s_2}{2\pi}\right)^3 - \left(\frac{s_1}{2\pi}\right)^3 \left(\frac{s_2}{2\pi}\right)^3} \end{array} \right)} \right)$$

iii) $$\chi(Ћ) = \left(\left[\sqrt[3]{1 - \left(1 - m^3\right)^{\chi}}, \sqrt[3]{1 - \left(1 - n^3\right)^{\chi}} \right] e^{i\left(\sqrt[3]{1 - \left(1 - \left(\frac{p}{2\pi}\right)^3\right)^{\chi}}, \sqrt[3]{1 - \left(1 - \left(\frac{q}{2\pi}\right)^3\right)^{\chi}} \right)}, \\ \left[d^{\chi}, g^{\chi}\right] e^{i\left(\left(\frac{r}{2\pi}\right)^{\chi}, \left(\frac{a}{2\pi}\right)^{\chi}\right)} \right)$$

iv) $$(Ћ)^{\chi} = \left(\begin{array}{l} \left[m^{\chi}, n^{\chi}\right] e^{i\left(\left(\frac{p}{2\pi}\right)^{\chi}, \left(\frac{q}{2\pi}\right)^{\chi}\right)}, \\ \left[\sqrt[3]{1-\left(1-d^{3}\right)^{\chi}}, \sqrt[3]{1-\left(1-g^{3}\right)^{\chi}}\right] e^{i\left(\sqrt[3]{1-\left(1-\left(\frac{r}{2\pi}\right)^{3}\right)^{\chi}}, \sqrt[3]{1-\left(1-\left(\frac{s}{2\pi}\right)^{3}\right)^{\chi}}\right)} \end{array} \right)$$

Definition 7: Let $Ћ = \left([m,n]e^{i(p,q)}, [d,g]e^{i(r,s)}\right)$ be a CIVFFN, then score valve defined as: $score(Ћ) = \frac{1}{2}\left((m^3 + n^3) - (d^3 + g^3) + \frac{1}{2\pi}(p^3 + q^3) - (r^3 + s^3)\right)$ with $scoe(Ћ) \in [-1,1]$.

Definition 8: Let $Ћ_j = \left([m_j, n_j]e^{i(p_j,q_j)}, [d_j, g_j]e^{i(r_j,s_j)}\right)(j = 1,2)$ be a collection of CIVFFNs, then the conditions are satisfied.

i) $$Ћ_1 \cup Ћ_2 = \left(\begin{array}{l} \left[\max\{m_1, m_2\}, \max\{n_1, n_2\}\right] e^{i\left[\max\{p_1, p_2\}, \max\{q_1, q_2\}\right]}, \\ \left[\min\{d_1, d_2\}, \min\{g_1, g_2\}\right] e^{i\left[\min\{r_1, r_2\}, \min\{s_1, s_2\}\right]} \end{array} \right)$$

ii) $$Ћ_1 \cap Ћ_2 = \left(\begin{array}{l} \left[\min\{m_1, m_2\}, \min\{n_1, n_2\}\right] e^{\left[\min\{p_1, p_2\}, \min\{q_1, q_2\}\right]} \\ \left[\max\{d_1, d_2\}, \max\{g_1, g_2\}\right] e^{\left[\max\{r_1, r_2\}, \max\{s_1, s_2\}\right]} \end{array} \right)$$

iii) $$(Ћ)^{c} = \left([d,g]e^{i(r,s)}, [m,n]e^{i(p,q)}\right)$$

Theorem 1: Let $Ћ_j = \left([m_j, n_j]e^{i(p_j,q_j)}, [d_j, g_j]e^{i(r_j,s_j)}\right)(j = 1,2)$ be a collection of CIVFFNs, and $\chi \succ 0$, then the conditions are satisfied.

i) $Ћ_1 \cup Ћ_2 = Ћ_2 \cup Ћ_1$

ii) $Ћ_1 \cap Ћ_2 = Ћ_2 \cap Ћ_1$

iii) $\chi(Ћ_1 \cup Ћ_2) = \chi(Ћ_1) \cup \chi(Ћ_2)$

iv) $\chi(Ћ_1 \cap Ћ_2) = \chi(Ћ_1) \cap \chi(Ћ_2)$

v) $(Ћ_1 \cup Ћ_2)^{\chi} = (Ћ_1)^{\chi} \cup (Ћ_2)^{\chi}$

vi) $(Ћ_1 \cap Ћ_2)^{\chi} = (Ћ_1)^{\chi} \cap (Ћ_2)^{\chi}$

Proof: We establish parts (i, ii), with the remaining parts proven similarly.

i) Since $Ћ_1$ and $Ћ_2$ are two CIVFFNs, then by using Definition 8, we have:

$$Ћ_1 \cup Ћ_2 = \left(\begin{array}{l} \left[\max\{m_1, m_2\}, \max\{n_1, n_2\}\right] e^{i2\pi\left[\max\{p_1, p_2\}, \max\{q_1, q_2\}\right]}, \\ \left[\min\{d_1, d_2\}, \min\{g_1, g_2\}\right] e^{i2\pi\left[\min\{r_1, r_2\}, \min\{s_1, s_2\}\right]} \end{array} \right)$$

$$= \left(\begin{array}{l} \left[\max\{m_2, m_1\}, \max\{n_2, n_1\}\right] e^{i2\pi\left[\max\{p_2, p_1\}, \max\{q_2, q_1\}\right]}, \\ \left[\min\{d_2, d_1\}, \min\{g_2, g_1\}\right] e^{i2\pi\left[\min\{r_2, r_1\}, \min\{s_2, s_1\}\right]} \end{array} \right)$$

$$= Ћ_2 \cup Ћ_1$$

ii) Again, by using Definition 8, we have:

$$Ь_1 \cap Ь_2 = \begin{pmatrix} \left[\min\{m_1, m_2\}, \min\{n_1, n_2\}\right] e^{i\left[\min\{p_1, p_2\}, \min\{q_1, q_2\}\right]}, \\ \left[\max\{d_1, d_2\}, \max\{g_1, g_2\}\right] e^{i\left[\max\{r_1, r_2\}, \max\{s_1, s_2\}\right]} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\min\{m_2, m_1\}, \min\{n_2, n_1\}\right] e^{i\left[\min\{p_2, p_1\}, \min\{q_2, q_1\}\right]}, \\ \left[\max\{d_2, d_1\}, \max\{g_2, g_1\}\right] e^{i\left[\max\{r_2, r_1\}, \max\{s_2, s_1\}\right]} \end{pmatrix}$$

$$= Ь_2 \cap Ь_1$$

Thus, the proof is concluded.

4. New Methods Based on Algebraic Operations

This section introduces complex approaches under interval-valued fuzzy information, including the CIVFFWA, CIVFFOWA, CIVFFWG, and CIVFFOWG operators. These methods are designed to process interval-valued fuzzy data, each providing distinct techniques for information aggregation. Additionally, their key structural properties, such as idempotency, boundedness, and monotonicity, are examined.

Definition 9: Let $Ь_j = \left([m_j, n_j]e^{i(p_j,q_j)}, [d_j, g_j]e^{i(r_j,s_j)}\right)(j = 1,2,\ldots,n)$ be a finite group of CIVFFNs, and let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ represent their weights satisfying the conditions: $\varphi_j \in [0,1]$ and $\sum_{j=1}^{n} \varphi_j = 1$, then the CIVFFWA operator is defined as:

$$\text{CIVFFWA}_{\varphi}\left(Ь_1, Ь_2, \ldots, Ь_n\right)$$

$$= \begin{pmatrix} \left[\sqrt[3]{1-\prod_{j=1}^{n}\left(1-m_j^3\right)^{\varphi_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-n_j^3\right)^{\varphi_j}}\right] e^{i\left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{p_j}{2\pi}\right)^3\right)^{\varphi_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{q_j}{2\pi}\right)^3\right)^{\varphi_j}}\right)}, \\ \left[\prod_{j=1}^{n}\left(d_j\right)^{\varphi_j}, \prod_{j=1}^{n}\left(g_j\right)^{\varphi_j}\right] e^{i\left(\prod_{j=1}^{n}\left(\frac{r_j}{2\pi}\right)^{\varphi_j}, \prod_{j=1}^{n}\left(\frac{s_j}{2\pi}\right)^{\varphi_j}\right)} \end{pmatrix}$$

Example 1: To develop the novel method, we consider an example with four CFFNs, such as: $Ь_1 = \left([0.6,0.7]e^{i[0.4,0.8]}, [0.5,0.8]e^{i[0.3,0.5]}\right)$, $Ь_2 = \left([0.6,0.8]e^{i[0.4,0.5]}, [0.4,0.5]e^{i[0.5,0.6]}\right)$, $Ь_3 = \left([0.4,0.5]e^{i[0.3,0.5]}, [0.5,0.6]e^{i[0.5,0.7]}\right)$, $Ь_4 = \left([0.5,0.6]e^{i[0.5,0.7]}, [0.3,0.5]e^{i[0.4,0.6]}\right)$ are four values with corresponding weighted vector $\varphi = (0.1,0.2,0.3,0.4)$, then we have:

$$\sqrt[3]{1-\prod_{j=1}^{4}\left(1-m_j^3\right)^{\varphi_j}} = \sqrt[3]{1-\left(1-(0.6)^3\right)^{0.1}\left(1-(0.6)^3\right)^{0.2}\left(1-(0.4)^3\right)^{0.3}\left(1-(0.5)^3\right)^{0.4}} = 0.51$$

$$\sqrt[3]{1-\prod_{j=1}^{4}\left(1-n_j^3\right)^{\varphi_j}} = \sqrt[3]{1-\left(1-(0.7)^3\right)^{0.1}\left(1-(0.8)^3\right)^{0.2}\left(1-(0.5)^3\right)^{0.3}\left(1-(0.6)^3\right)^{0.4}} = 0.65$$

$$\sqrt[3]{1-\prod_{j=1}^{4}\left(1-\left(\frac{p_j}{2\pi}\right)^3\right)^{\varphi_j}}=\sqrt[3]{1-\left(1-\left(\frac{0.4}{2\pi}\right)^3\right)^{0.1}\left(1-\left(\frac{0.4}{2\pi}\right)^3\right)^{0.2}\left(1-\left(\frac{0.3}{2\pi}\right)^3\right)^{0.3}\left(1-\left(\frac{0.5}{2\pi}\right)^3\right)^{0.4}}=0.42$$

$$\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{q_j}{2\pi}\right)^3\right)^{\varphi_j}}=\sqrt[3]{1-\left(1-\left(\frac{0.8}{2\pi}\right)^3\right)^{0.1}\left(1-\left(\frac{0.5}{2\pi}\right)^3\right)^{0.2}\left(1-\left(\frac{0.5}{2\pi}\right)^3\right)^{0.3}\left(1-\left(\frac{0.7}{2\pi}\right)^3\right)^{0.4}}=0.64$$

$$\prod_{j=1}^{4}\left(d_j\right)^{\varphi_j}=(0.5)^{0.1}(0.4)^{0.2}(0.5)^{0.3}(0.3)^{0.4}=0.38$$

$$\prod_{j=1}^{4}\left(g_j\right)^{\varphi_j}=(0.8)^{0.1}(0.5)^{0.2}(0.6)^{0.3}(0.5)^{0.4}=0.55$$

$$\prod_{j=1}^{4}\left(\frac{r_j}{2\pi}\right)^{\varphi_j}=\left(\frac{0.3}{2\pi}\right)^{0.1}\left(\frac{0.5}{2\pi}\right)^{0.2}\left(\frac{0.5}{2\pi}\right)^{0.3}\left(\frac{0.4}{2\pi}\right)^{0.4}=0.43$$

$$\prod_{j=1}^{4}\left(\frac{s_j}{2\pi}\right)^{\varphi_j}=\left(\frac{0.5}{2\pi}\right)^{0.1}\left(\frac{0.6}{2\pi}\right)^{0.2}\left(\frac{0.7}{2\pi}\right)^{0.3}\left(\frac{0.5}{2\pi}\right)^{0.4}=0.56$$

Using the Definition 9, we have $\text{CIVFFWA}_{\varphi}(\text{Ѣ}_1,\text{Ѣ}_2,\ldots,\text{Ѣ}_4)=\left([0.51,0.65]e^{i[0.42,0.64]},[0.38,0.55]e^{i[0.43,0.56]}\right)$.

Theorem 2: Let $\text{Ѣ}_j=\left(\left[m_j,n_j\right]e^{i(p_j,q_j)},\left[d_j,g_j\right]e^{i(r_j,s_j)}\right)(1\le j\le n)$ be a group of CFFVs, then their resulting value under CIVFFWA operator is still a CFFV.

$$\text{CIVFFWA}_{\varphi}\left(\text{Ѣ}_1,\text{Ѣ}_2,\ldots,\text{Ѣ}_n\right)$$

$$=\left(\begin{array}{l}\left[\begin{array}{l}\sqrt[3]{1-\prod_{j=1}^{n}\left(1-m_j^3\right)^{\varphi_j}},\\ \sqrt[3]{1-\prod_{j=1}^{n}\left(1-n_j^3\right)^{\varphi_j}}\end{array}\right]e^{i\left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{p_j}{2\pi}\right)^3\right)^{\varphi_j}},\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{q_j}{2\pi}\right)^3\right)^{\varphi_j}}\right)},\\ \left[\prod_{j=1}^{n}\left(d_j\right)^{\varphi_j},\prod_{j=1}^{n}\left(g_j\right)^{\varphi_j}\right]e^{i\left(\prod_{j=1}^{n}\left(\frac{r_j}{2\pi}\right)^{\varphi_j},\prod_{j=1}^{n}\left(\frac{s_j}{2\pi}\right)^{\varphi_j}\right)}\end{array}\right)$$

Proof: By the principle of mathematical induction, we outline the following major steps:

Step 1: For $n=2$, then we have:

$$\varphi_1(\mathcal{H}_1)=\left(\begin{array}{l}\left[\sqrt[3]{1-\left(1-m_1^3\right)^{\varphi_1}},\sqrt[3]{1-\left(1-n_1^3\right)^{\varphi_1}}\right]e^{i\left(\sqrt[3]{1-\left(1-\left(\frac{p_1}{2\pi}\right)^3\right)^{\varphi_1}},\sqrt[3]{1-\left(1-\left(\frac{q_1}{2\pi}\right)^3\right)^{\varphi_1}}\right)},\\ \left[(d_1)^{\varphi_1},(g_1)^{\varphi_1}\right]e^{i\left[\left(\frac{r_1}{2\pi}\right)^{\varphi_1},\left(\frac{s_1}{2\pi}\right)^{\varphi_1}\right]}\end{array}\right)$$

$$\varphi_2(\mathcal{H}_2)=\left(\begin{array}{l}\left[\sqrt[3]{1-\left(1-m_2^3\right)^{\varphi_2}},\sqrt[3]{1-\left(1-n_2^3\right)^{\varphi_2}}\right]e^{i\left(\sqrt[3]{1-\left(1-\left(\frac{p_2}{2\pi}\right)^3\right)^{\varphi_2}},\sqrt[3]{1-\left(1-\left(\frac{q_2}{2\pi}\right)^3\right)^{\varphi_2}}\right)},\\ \left[(d_2)^{\varphi_2},(g_2)^{\varphi_2}\right]e^{i\left[\left(\frac{r_2}{2\pi}\right)^{\varphi_2},\left(\frac{s_1}{2\pi}\right)^{\varphi_2}\right]}\end{array}\right)$$

By Definition 9, we have:

$$\mathrm{CIVFFWA}_{\varphi}(\mathcal{H}_1,\mathcal{H}_2)$$

$$=\left(\begin{array}{l}\left[\begin{array}{l}\sqrt[3]{1-\prod_{j=1}^{2}\left(1-m_j^3\right)^{\varphi_j}},\\ \sqrt[3]{1-\prod_{j=1}^{2}\left(1-n_j^3\right)^{\varphi_j}}\end{array}\right]e^{i\left(\sqrt[3]{1-\prod_{j=1}^{2}\left(1-\left(\frac{p_j}{2\pi}\right)^3\right)^{\varphi_j}},\sqrt[3]{1-\prod_{j=1}^{2}\left(1-\left(\frac{q_j}{2\pi}\right)^3\right)^{\varphi_j}}\right)},\\ \left[\prod_{j=1}^{2}\left(d_j\right)^{\varphi_j},\prod_{j=1}^{2}\left(g_j\right)^{\varphi_j}\right]e^{i\left(\prod_{j=1}^{2}\left(\frac{r_j}{2\pi}\right)^{\varphi_j},\prod_{j=1}^{2}\left(\frac{s_j}{2\pi}\right)^{\varphi_j}\right)}\end{array}\right)$$

Step 2: It holds for $n=2$. Next assuming that it is holds for $n=k$, $k\succ 0$, then we have:

$$\mathrm{CIVFFWA}_{\varphi}(\mathcal{H}_1,\mathcal{H}_2,\dots,\mathcal{H}_k)$$

$$=\left(\begin{array}{l}\left[\begin{array}{l}\sqrt[3]{1-\prod_{j=1}^{k}\left(1-m_j^3\right)^{\varphi_j}},\\ \sqrt[3]{1-\prod_{j=1}^{k}\left(1-n_j^3\right)^{\varphi_j}}\end{array}\right]e^{i\left(\sqrt[3]{1-\prod_{j=1}^{k}\left(1-\left(\frac{p_j}{2\pi}\right)^3\right)^{\varphi_j}},\sqrt[3]{1-\prod_{j=1}^{k}\left(1-\left(\frac{q_j}{2\pi}\right)^3\right)^{\varphi_j}}\right)},\\ \left[\prod_{j=1}^{k}\left(d_j\right)^{\varphi_j},\prod_{j=1}^{k}\left(g_j\right)^{\varphi_j}\right]e^{i\left(\prod_{j=1}^{k}\left(\frac{r_j}{2\pi}\right)^{\varphi_j},\prod_{j=1}^{k}\left(\frac{s_j}{2\pi}\right)^{\varphi_j}\right)}\end{array}\right)$$

Step 3: If the given result holds for $n=k$, next we show that it is true for $n=k+1$.

$$\mathrm{CIVFFWA}_{\varphi}\left(\text{Њ}_1,\text{Њ}_2,\ldots,\text{Њ}_k,\text{Њ}_{k+1}\right)$$

$$=\left(\begin{array}{l}\left[\sqrt[3]{1-\prod_{j=1}^{k}\left(1-m_j^3\right)^{\varphi_j}},\ \sqrt[3]{1-\prod_{j=1}^{k}\left(1-n_j^3\right)^{\varphi_j}}\right]e^{i\left(\sqrt[3]{1-\prod_{j=1}^{k}\left(1-\left(\frac{p_j}{2\pi}\right)^3\right)^{\varphi_j}},\ \sqrt[3]{1-\prod_{j=1}^{k}\left(1-\left(\frac{q_j}{2\pi}\right)^3\right)^{\varphi_j}}\right)},\\ \left[\prod_{j=1}^{k}\left(d_j\right)^{\varphi_j},\ \prod_{j=1}^{k}\left(g_j\right)^{\varphi_j}\right]e^{i\left(\prod_{j=1}^{k}\left(\frac{r_j}{2\pi}\right)^{\varphi_j},\ \prod_{j=1}^{k}\left(\frac{s_j}{2\pi}\right)^{\varphi_j}\right)}\end{array}\right)\oplus$$

$$\left(\begin{array}{l}\left[\sqrt[3]{1-\left(1-m_{k+1}^3\right)^{\varphi_{k+1}}},\ \sqrt[3]{1-\left(1-n_{k+1}^3\right)^{\varphi_{k+1}}}\right]e^{i\left(\sqrt[3]{1-\left(1-\left(\frac{p_{k+1}}{2\pi}\right)^3\right)^{\varphi_{k+1}}},\ \sqrt[3]{1-\left(1-\left(\frac{q_{k+1}}{2\pi}\right)^3\right)^{\varphi_{k+1}}}\right)},\\ \left[\left(d_{k+1}\right)^{\varphi_{k+1}},\left(g_{k+1}\right)^{\varphi_{k+1}}\right]e^{i\left(\left(\frac{r_{k+1}}{2\pi}\right)^{\varphi_{k+1}},\left(\frac{s_j}{2\pi}\right)^{\varphi_{k+1}}\right)}\end{array}\right)$$

$$=\left(\begin{array}{l}\left[\sqrt[3]{1-\prod_{j=1}^{k+1}\left(1-m_j^3\right)^{\varphi_j}},\ \sqrt[3]{1-\prod_{j=1}^{k+1}\left(1-n_j^3\right)^{\varphi_j}}\right]e^{i\left(\sqrt[3]{1-\prod_{j=1}^{k+1}\left(1-\left(\frac{p_j}{2\pi}\right)^3\right)^{\varphi_j}},\ \sqrt[3]{1-\prod_{j=1}^{k+1}\left(1-\left(\frac{q_j}{2\pi}\right)^3\right)^{\varphi_j}}\right)},\\ \left[\prod_{j=1}^{k+1}\left(d_j\right)^{\varphi_j},\ \prod_{j=1}^{k+1}\left(g_j\right)^{\varphi_j}\right]e^{i\left(\prod_{j=1}^{k+1}\left(\frac{r_j}{2\pi}\right)^{\varphi_j},\ \prod_{j=1}^{k+1}\left(\frac{s_j}{2\pi}\right)^{\varphi_j}\right)}\end{array}\right)$$

It is true for $n = k + 1$. Thus, it is true for all positive integers *n*.

Idempotency: Let $\text{Њ}_j = \left(\left[m_j, n_j\right]e^{i\left(p_j,q_j\right)}, \left[d_j, g_j\right]e^{i\left(r_j,s_j\right)}\right)(1 \le j \le n)$ be a group of CIVFFNs, such that $\text{Њ}_j\left((1 \le j \le n)\right) = \text{Њ}$ for all j, then:

$\text{CIVFFWA}_{\varphi}(Њ_1, Њ_2, \ldots, Њ_n) = Њ$ (2)

Boundedness: Let $Њ_j = \left([m_j, n_j]e^{i(p_j, q_j)}, [d_j, g_j]e^{i(r_j, s_j)}\right)(1 \leq j \leq n)$ be a finite group of CIVFFNs, then:

$Њ^- \leq \text{CIVFFWA}_{\varphi}(Њ_1, Њ_2, \ldots, Њ_n) \leq Њ^+$ (3)

Where, $Њ^-$ and $Њ^+$ are called the minimum and maximum values.

Monotonicity: Let two group $Њ_j = \left([m_j, n_j]e^{i(p_j, q_j)}, [d_j, g_j]e^{i(r_j, s_j)}\right)(1 \leq j \leq n)$ and $Њ_j^* = \left([m_j^*, n_j^*]e^{i\left(p_j^*, q_j^*\right)}, [d_j^*, g_j^*]e^{i\left(r_j^*, s_j^*\right)}\right)$ $(1 \leq j \leq n)$ of CIVFFNs, under condition: $m_j \leq m_j^*$, $n_j \leq n_j^*$, $p_j \leq p_j^*$, $q_j \leq q_j^*$, $d_j \geq d_j^*$, $g_j \geq g_j^*$, $r_j \geq r_j^*$, $s_j \geq s_j^*$, then:

$\text{CIVFFWA}_{\varphi}(Њ_1, Њ_1, \ldots, Њ_n) \leq \text{CIVFFWA}_{\varphi}(Њ_1^*, Њ_2^*, \ldots, Њ_n^*)$ (4)

Definition 10: Let $Њ_j = \left([m_j, n_j]e^{i(p_j, q_j)}, [d_j, g_j]e^{i(r_j, s_j)}\right)(1 \leq j \leq n)$ be a collection of CIVFFNs, and let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ represent their weights satisfying the conditions: $\varphi_j \in [0,1]$ and $\sum_{j=1}^{n} \varphi_j = 1$. And $\left(\sigma(1), \sigma(2), \ldots, \sigma(n)\right)$ be any permutation of $(1,2,\ldots,n)$, with $Њ_{\sigma(j)} \leq Њ_{\sigma(j-1)}$, then the CIVFFOWA operator is defined mathematically as:

$$\text{CIVFFOWA}_{\varphi}\left(Њ_1, Њ_2, \ldots, Њ_n\right) = \left(\begin{array}{l} \left[\begin{array}{l} \sqrt[3]{1 - \prod_{j=1}^{n}\left(1 - m_{\sigma(j)}^3\right)^{\varphi_j}}, \\ \sqrt[3]{1 - \prod_{j=1}^{n}\left(1 - n_{\sigma(j)}^3\right)^{\varphi_j}} \end{array} \right] e^{i\left(\sqrt[3]{1 - \prod_{j=1}^{n}\left(1 - \left(\frac{p_{\sigma(j)}}{2\pi}\right)^3\right)^{\varphi_j}}, \sqrt[3]{1 - \prod_{j=1}^{n}\left(1 - \left(\frac{q_{\sigma(j)}}{2\pi}\right)^3\right)^{\varphi_j}} \right)}, \\ \left[\prod_{j=1}^{n}\left(d_{\sigma(j)}\right)^{\varphi_j}, \prod_{j=1}^{n}\left(g_{\sigma(j)}\right)^{\varphi_j} \right] e^{i\left(\prod_{j=1}^{n}\left(\frac{r_{\sigma(j)}}{2\pi}\right)^{\varphi_j}, \prod_{j=1}^{n}\left(\frac{s_{\sigma(j)}}{2\pi}\right)^{\varphi_j} \right)} \end{array} \right)$$

Definition 11: Let $Њ_j = \left([m_j, n_j]e^{i(p_j, q_j)}, [d_j, g_j]e^{i(r_j, s_j)}\right)(1 \leq j \leq n)$ be a finite group of CIVFFNs, and let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ represent their weights satisfying the conditions: $\varphi_j \in [0,1]$ and $\sum_{j=1}^{n} \varphi_j = 1$, then the CIVFFWG operator is defined as:

$$\text{CIVFFWG}_{\varphi}\left(Њ_1, Њ_2, \ldots, Њ_n\right) = \left(\begin{array}{l} \left[\prod_{j=1}^{n}\left(m_j\right)^{\varphi_j}, \prod_{j=1}^{n}\left(n_j\right)^{\varphi_j} \right] e^{i\left(\prod_{j=1}^{n}\left(\frac{p_j}{2\pi}\right)^{\varphi_j}, \prod_{j=1}^{n}\left(\frac{q_j}{2\pi}\right)^{\varphi_j} \right)}, \\ \left[\begin{array}{l} \sqrt[3]{1 - \prod_{j=1}^{n}\left(1 - d_j^3\right)^{\varphi_j}}, \\ \sqrt[3]{1 - \prod_{j=1}^{n}\left(1 - g_j^3\right)^{\varphi_j}} \end{array} \right] e^{i\left(\sqrt[3]{1 - \prod_{j=1}^{n}\left(1 - \left(\frac{r_j}{2\pi}\right)^3\right)^{\varphi_j}}, \sqrt[3]{1 - \prod_{j=1}^{n}\left(1 - \left(\frac{s_j}{2\pi}\right)^3\right)^{\varphi_j}} \right)} \end{array} \right)$$

Theorem 3: Let $Ҍ_j = \left([m_j, n_j]e^{i(p_j,q_j)}, [d_j, g_j]e^{i(r_j,s_j)}\right)(1 \le j \le n)$ be a group of CFFVs, then their resulting value under CIVFFWG operator is still a CFFV.

Proof: For proof see the proof of Theorem 2.

Idempotency: Let $Ҍ_j = \left([m_j, n_j]e^{i(p_j,q_j)}, [d_j, g_j]e^{i(r_j,s_j)}\right)(1 \le j \le n)$ be a group of CIVFFNs, such that $Ҍ_j\left((1 \le j \le n)\right) = Ҍ$ for all j, then:

$$\text{CIVFFWG}_{\varphi}(Ҍ_1, Ҍ_2, \ldots, Ҍ_n) = Ҍ \quad (5)$$

Boundedness: Let $Ҍ_j = \left([m_j, n_j]e^{i(p_j,q_j)}, [d_j, g_j]e^{i(r_j,s_j)}\right)(1 \le j \le n)$ be a finite group of CIVFFNs, then:

$$Ҍ^- \le \text{CIVFFWG}_{\varphi}(Ҍ_1, Ҍ_2, \ldots, Ҍ_n) \le Ҍ^+ \quad (6)$$

Where, $Ҍ^-$ and $Ҍ^+$ are called the minimum and maximum values.

Monotonicity: Let two group $Ҍ_j = \left([m_j, n_j]e^{i(p_j,q_j)}, [d_j, g_j]e^{i(r_j,s_j)}\right)(1 \le j \le n)$ and $Ҍ_j^* = \left([m_j^*, n_j^*]e^{i(p_j^*,q_j^*)}, [d_j^*, g_j^*]e^{i(r_j^*,s_j^*)}\right)$ $(1 \le j \le n)$ of CIVFFNs, under condition:$m_j \le m_j^*$, $n_j \le n_j^*, p_j \le p_j^*, q_j \le q_j^*$, $d_j \ge d_j^*, g_j \ge g_j^*, r_j \ge r_j^*, s_j \ge s_j^*$, then:

$$\text{CIVFFWG}_{\varphi}(Ҍ_1, Ҍ_1, \ldots, Ҍ_n) \le \text{CIVFFWG}_{\varphi}(Ҍ_1^*, Ҍ_2^*, \ldots, Ҍ_n^*) \quad (7)$$

Definition 12: Let $Ҍ_j = \left([m_j, n_j]e^{i(p_j,q_j)}, [d_j, g_j]e^{i(r_j,s_j)}\right)(1 \le j \le n)$ be a collection of CIVFFNs, and let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ represent their weights satisfying the conditions: $\varphi_j \in [0,1]$ and $\sum_{j=1}^{n} \varphi_j = 1$. And $\left(\sigma(1), \sigma(2), \ldots, \sigma(n)\right)$ be any permutation of $(1,2,\ldots,n)$, with $Ҍ_{\sigma(j)} \le Ҍ_{\sigma(j-1)}$, then the CIVFFOWG operator is defined mathematically as:

$$\text{CIVFFOWG}_{\varphi}\left(Ҍ_1, Ҍ_2, \ldots, Ҍ_n\right) = \left(\begin{array}{l} \left[\prod_{j=1}^{n}\left(m_{\sigma(j)}\right)^{\varphi_j}, \prod_{j=1}^{n}\left(n_{\sigma(j)}\right)^{\varphi_j}\right] e^{i\left(\prod_{j=1}^{n}\left(\frac{p_{\sigma(j)}}{2\pi}\right)^{\varphi_j}, \prod_{j=1}^{n}\left(\frac{q_{\sigma(j)}}{2\pi}\right)^{\varphi_j}\right)}, \\ \left[\begin{array}{l}\sqrt[3]{1-\prod_{j=1}^{n}\left(1-d_{\sigma(j)}^3\right)^{\varphi_j}}, \\ \sqrt[3]{1-\prod_{j=1}^{n}\left(1-g_{\sigma(j)}^3\right)^{\varphi_j}}\end{array}\right] e^{i\left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{r_{\sigma(j)}}{2\pi}\right)^3\right)^{\varphi_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{s_{\sigma(j)}}{2\pi}\right)^3\right)^{\varphi_j}}\right)} \end{array} \right)$$

5. Algorithmic Selection of the Best Classroom Teaching Mode

The multi-criteria group decision-making (MCGDM) process is employed to identify the most suitable option from a set of alternatives based on expert evaluations. A panel of specialists assesses each alternative using predefined criteria, assigning scores that reflect how well each option meets these standards. These scores are weighted according to the significance of each attribute, forming decision matrices that facilitate the overall evaluation. The experts' assessments are then aggregated to determine the optimal choice, ensuring well-informed decision-making by incorporating multiple factors. MCGDM is widely applied across various domains, including business, engineering, and economics, where decisions require thorough analysis of multiple aspects. By integrating diverse expert perspectives, it enhances decision accuracy and balance. This approach is particularly beneficial when numerous alternatives exist, each with distinct characteristics. Ultimately, MCGDM aids decision-makers in selecting the most appropriate option by leveraging expert insights and structured evaluation criteria.

Algorithm: The process of selecting the most suitable teaching mode in a university involves analysing a list of teaching methods, represented as:$Å = \{Å_1, Å_2, \dots, Å_m\}$ and evaluating them based on various factors $C = \{C_1, C_2, \dots, C_n\}$ is a list of the factors considered when evaluating class room teaching modes. Each factor is given a specific weight, $\varphi = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$, reflecting its importance in the decision-making process. A team of experts $E = \{E_1, E_2, \dots, E_k\}$ assesses each teaching mode based on these factors. Additionally, each expert's input is weighted by their significance, represented as $\omega = \{\omega_1, \omega_2, \dots, \omega_k\}$. The objective is to determine the most appropriate teaching mode by evaluating both the significance of various factors and the expertise of the decision-making team. This approach ensures a well-balanced and informed selection, aligning the chosen teaching method with students' learning needs for optimal outcomes.

Step 1 – Decision-makers' insights can be structured into a matrix that evaluates different alternatives based on key attributes. This comparative approach facilitates a clearer understanding of each option, enabling a more informed and efficient selection of the most suitable alternative.

$$E^k = \begin{array}{c} \\ Å_1 \\ Å_2 \\ \cdot \\ \cdot \\ \cdot \\ Å_m \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & \cdots & C_n \end{array} \\ \begin{pmatrix} Ћ_{11}^{(k)} & Ћ_{12}^{(k)} & Ћ_{13}^{(k)} & \cdots & Ћ_{1n}^{(k)} \\ Ћ_{21}^{(k)} & Ћ_{22}^{(k)} & Ћ_{23}^{(k)} & \cdots & Ћ_{2n}^{(k)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ Ћ_{m1}^{(k)} & Ћ_{m2}^{(k)} & Ћ_{m3}^{(k)} & \cdots & Ћ_{mn}^{(k)} \end{pmatrix} \end{array}$$

Step 2 – Combine all individual CIVFF decision matrices, such as $E^t (1 \leq t \leq k)$ into a single collective CIVIF decision matrix $E = (Ћ_{ij})_{mn}$ using the CIVFFWA and CIVFFWG operators with weights $\omega = \{\omega_1, \omega_2, \dots, \omega_k\}$ for a unified group decision.

Step *3* – Again using the CIVFFWA and CIVFFWG operators with weights $\varphi = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$, the all preference values $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ are calculated from the collective decision matrix. These methods ensure a thorough evaluation by considering multiple factors systematically.

Step 4 – Calculating the score functions by using Definition 7, of all preference values $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ entails assigning numerical assessments to each preference, facilitating precise evaluation within a specific context.

Step 5 – Rank the options based on their scores values, with the highest ones being the most suitable and desirable. Select the best alternative based on the highest-ranking value.

Figure 1 provides a visual representation of the step-by-step process for determining the most appropriate teaching mode, taking into account various factors, expert assessments, and their corresponding weights.

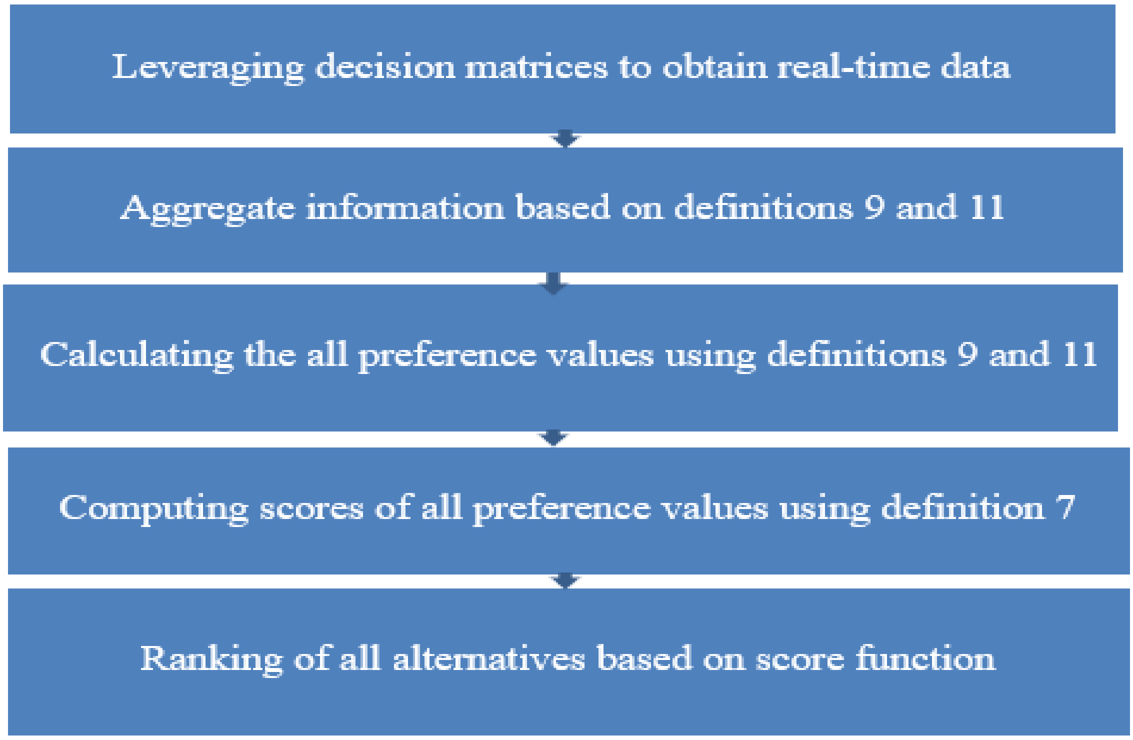


Fig.1. Decision-Making Process for Optimal Teaching Mode Selection

6. Illustrative Example

Selecting appropriate teaching methods is essential for enhancing students' learning outcomes. Effective instructional strategies enable educators to cater to diverse student needs, fostering engagement and motivation within the classroom. Consequently, this contributes to improved academic performance and a more enriching learning experience.

Case Study: Classroom Teaching Model Selection: University teaching modes vary based on different approaches, including student engagement. However, management often selects methods without considering key factors, affecting effectiveness. Common teaching modes in university education are discussed below.

$\mathring{A}_1$: *Lecture-Based Teaching (LBT):* Instructors deliver content in a structured format, focusing on knowledge transfer through verbal explanations.

$\mathring{A}_2$: *Collaborative Learning Method (CLM):* Students work together in groups, sharing ideas and solving problems, fostering teamwork and peer learning.

$\mathring{A}_3$: *Project-Based Learning (PBL)*: Students engage in extended projects that encourage problem-solving and application of knowledge to real-world scenarios.

$\mathring{A}_4$: *Case-Based Learning (CBM):* Students analyse and discuss real-life cases to develop problem-solving and decision-making skills.

Different institutions adopt various teaching methods, each with a unique approach. This study evaluates these methods based on key attributes essential for enhancing student learning. While multiple factors influence effectiveness, we focus on the following four:

C_1: *Student Engagement (SE):* The method should promote active student engagement and participation, maintaining interest and motivation throughout the lesson.

C_2: *Learning Objectives (LO):* The method should align with the lesson's specific learning outcomes, ensuring the development of the intended skills, knowledge, or competencies.

C_3: ***Student Centred (SC):*** Effective teaching methods should prioritise learners' needs, interests, and abilities, encouraging active participation in the learning process.

C_4: ***Assessment Compatibility:*** The method should support formative, diagnostic, and summative assessments, enabling continuous evaluation of students' understanding and progress.

The method should align with lesson objectives, ensuring students acquire essential skills and knowledge. It must actively engage students, maintaining motivation and participation. Additionally, it should be adaptable to diverse student needs and classroom settings while remaining feasible within available resources for effective learning. In this analysis, classroom teaching models $\{\mathring{A}_1, \mathring{A}_2, \mathring{A}_3, \mathring{A}_4\}$ are assessed based on the criteria $\{C_1, C_2, C_3, C_4\}$ with the guidance of experts $\{E_1, E_2, E_3, E_4\}$. Experts assess the effectiveness of each teaching model based on specific criteria to determine the most suitable approach for student learning. The expert evaluations are summarised in Tables 1–4.

Table 1
Assessment of Expert E_1

	C_1	C_2	C_3	C_4
A_1	$\left([0.64,0.75]e^{i2\pi[0.56,0.75]}, [0.57,0.78]e^{i2\pi[0.52,0.57]}\right)$	$\left([0.63,0.72]e^{i2\pi[0.56,0.68]}, [0.55,0.73]e^{i2\pi[0.48,0.57]}\right)$	$\left([0.63,0.72]e^{i2\pi[0.56,0.68]}, [0.55,0.73]e^{i2\pi[0.48,0.57]}\right)$	$\left([0.64,0.75]e^{i2\pi[0.56,0.75]}, [0.57,0.78]e^{i2\pi[0.52,0.57]}\right)$
A_2	$\left([0.64,0.67]e^{i2\pi[0.56,0.65]}, [0.54,0.67]e^{i2\pi[0.54,0.56]}\right)$	$\left([0.55,0.62]e^{i2\pi[0.54,0.70]}, [0.58,0.69]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.45,0.66]e^{i2\pi[0.54,0.70]}, [0.48,0.59]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.64,0.67]e^{i2\pi[0.56,0.65]}, [0.54,0.67]e^{i2\pi[0.54,0.56]}\right)$
A_3	$\left([0.58,0.69]e^{i2\pi[0.54,0.70]}, [0.55,0.62]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.55,0.66]e^{i2\pi[0.55,0.72]}, [0.61,0.67]e^{i2\pi[0.47,0.58]}\right)$	$\left([0.55,0.62]e^{i2\pi[0.54,0.70]}, [0.58,0.69]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.58,0.69]e^{i2\pi[0.54,0.70]}, [0.55,0.62]e^{i2\pi[0.45,0.58]}\right)$
A_4	$\left([0.65,0.73]e^{i2\pi[0.56,0.75]}, [0.56,0.76]e^{i2\pi[0.52,0.57]}\right)$	$\left([0.54,0.76]e^{i2\pi[0.63,0.65]}, [0.46,0.68]e^{i2\pi[0.54,0.67]}\right)$	$\left([0.54,0.76]e^{i2\pi[0.63,0.65]}, [0.46,0.68]e^{i2\pi[0.54,0.67]}\right)$	$\left([0.65,0.73]e^{i2\pi[0.56,0.75]}, [0.56,0.76]e^{i2\pi[0.52,0.57]}\right)$

Table 2
Assessment of Expert E_2

	C_1	C_2	C_3	C_4
A_1	$\left([0.62,0.67]e^{i2\pi[0.65,0.67]}, [0.61,0.68]e^{i2\pi[0.64,0.77]}\right)$	$\left([0.48,0.59]e^{i2\pi[0.54,0.70]}, [0.45,0.66]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.47,0.58]e^{i2\pi[0.55,0.72]}, [0.61,0.67]e^{i2\pi[0.55,0.66]}\right)$	$\left([0.54,0.82]e^{i2\pi[0.63,0.68]}, [0.47,0.67]e^{i2\pi[0.54,0.67]}\right)$
A_2	$\left([0.52,0.72]e^{i2\pi[0.54,0.65]}, [0.55,0.74]e^{i2\pi[0.46,0.65]}\right)$	$\left([0.61,0.68]e^{i2\pi[0.65,0.67]}, [0.62,0.67]e^{i2\pi[0.64,0.77]}\right)$	$\left([0.54,0.76]e^{i2\pi[0.63,0.65]}, [0.46,0.68]e^{i2\pi[0.54,0.67]}\right)$	$\left([0.65,0.73]e^{i2\pi[0.56,0.75]}, [0.56,0.76]e^{i2\pi[0.52,0.57]}\right)$
A_3	$\left([0.63,0.72]e^{i2\pi[0.56,0.68]}, [0.55,0.73]e^{i2\pi[0.48,0.57]}\right)$	$\left([0.54,0.65]e^{i2\pi[0.52,0.72]}, [0.56,0.74]e^{i2\pi[0.46,0.65]}\right)$	$\left([0.55,0.62]e^{i2\pi[0.54,0.70]}, [0.58,0.69]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.57,0.75]e^{i2\pi[0.52,0.75]}, [0.64,0.78]e^{i2\pi[0.56,0.57]}\right)$
A_4	$\left([0.55,0.62]e^{i2\pi[0.56,0.70]}, [0.53,0.73]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.64,0.77]e^{i2\pi[0.65,0.67]}, [0.62,0.67]e^{i2\pi[0.61,0.68]}\right)$	$\left([0.53,0.82]e^{i2\pi[0.65,0.68]}, [0.46,0.67]e^{i2\pi[0.54,0.67]}\right)$	$\left([0.54,0.73]e^{i2\pi[0.45,0.66]}, [0.52,0.59]e^{i2\pi[0.45,0.58]}\right)$

Table 3
Assessment of Expert E_3

	C_1	C_2	C_3	C_4
A_1	$\left([0.65,0.68]e^{i2\pi[0.56,0.65]}, [0.54,0.67]e^{i2\pi[0.54,0.56]}\right)$	$\left([0.55,0.62]e^{i2\pi[0.54,0.70]}, [0.45,0.58]e^{i2\pi[0.58,0.69]}\right)$	$\left([0.61,0.68]e^{i2\pi[0.65,0.67]}, [0.62,0.67]e^{i2\pi[0.64,0.77]}\right)$	$\left([0.65,0.68]e^{i2\pi[0.56,0.65]}, [0.54,0.67]e^{i2\pi[0.54,0.56]}\right)$

A_2	$\left([0.63,0.72]e^{i2\pi[0.56,0.68]}, [0.55,0.73]e^{i2\pi[0.48,0.57]}\right)$	$\left([0.64,0.67]e^{i2\pi[0.56,0.65]}, [0.54,0.67]e^{i2\pi[0.54,0.56]}\right)$	$\left([0.54,0.70]e^{i2\pi[0.45,0.66]}, [0.51,0.59]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.54,0.76]e^{i2\pi[0.63,0.65]}, [0.46,0.68]e^{i2\pi[0.54,0.67]}\right)$
A_3	$\left([0.54,0.65]e^{i2\pi[0.52,0.72]}, [0.56,0.74]e^{i2\pi[0.46,0.65]}\right)$	$\left([0.54,0.65]e^{i2\pi[0.52,0.72]}, [0.56,0.74]e^{i2\pi[0.46,0.65]}\right)$	$\left([0.58,0.69]e^{i2\pi[0.54,0.70]}, [0.55,0.62]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.53,0.82]e^{i2\pi[0.65,0.68]}, [0.46,0.67]e^{i2\pi[0.54,0.67]}\right)$
A_4	$\left([0.57,0.75]e^{i2\pi[0.52,0.75]}, [0.64,0.78]e^{i2\pi[0.56,0.57]}\right)$	$\left([0.55,0.66]e^{i2\pi[0.55,0.72]}, [0.61,0.67]e^{i2\pi[0.47,0.58]}\right)$	$\left([0.64,0.77]e^{i2\pi[0.65,0.67]}, [0.62,0.67]e^{i2\pi[0.61,0.68]}\right)$	$\left([0.54,0.76]e^{i2\pi[0.63,0.65]}, [0.46,0.68]e^{i2\pi[0.54,0.67]}\right)$

Table 4
Assessment of Expert E_4

	C_1	C_2	C_3	C_4
A_1	$\left([0.58,0.69]e^{i2\pi[0.54,0.70]}, [0.55,0.62]e^{i2\pi[0.45,0.58]}\right)$	$\left([0.55,0.75]e^{i2\pi[0.67,0.81]}, [0.54,0.64]e^{i2\pi[0.64,0.70]}\right)$	$\left([0.45,0.74]e^{i2\pi[0.54,0.84]}, [0.59,0.72]e^{i2\pi[0.45,0.53]}\right)$	$\left([0.64,0.77]e^{i2\pi[0.65,0.67]}, [0.62,0.67]e^{i2\pi[0.61,0.68]}\right)$
A_2	$\left([0.57,0.75]e^{i2\pi[0.52,0.75]}, [0.64,0.78]e^{i2\pi[0.56,0.57]}\right)$	$\left([0.54,0.76]e^{i2\pi[0.63,0.65]}, [0.54,0.67]e^{i2\pi[0.46,0.68]}\right)$	$\left([0.65,0.67]e^{i2\pi[0.56,0.65]}, [0.56,0.69]e^{i2\pi[0.54,0.56]}\right)$	$\left([0.48,0.59]e^{i2\pi[0.54,0.70]}, [0.45,0.66]e^{i2\pi[0.45,0.58]}\right)$
A_3	$\left([0.45,0.58]e^{i2\pi[0.54,0.70]}, [0.45,0.66]e^{i2\pi[0.48,0.59]}\right)$	$\left([0.64,0.78]e^{i2\pi[0.52,0.75]}, [0.57,0.75]e^{i2\pi[0.56,0.57]}\right)$	$\left([0.54,0.56]e^{i2\pi[0.56,0.65]}, [0.57,0.69]e^{i2\pi[0.65,0.67]}\right)$	$\left([0.54,0.70]e^{i2\pi[0.45,0.66]}, [0.51,0.59]e^{i2\pi[0.45,0.58]}\right)$
A_4	$\left([0.61,0.68]e^{i2\pi[0.65,0.67]}, [0.62,0.67]e^{i2\pi[0.64,0.77]}\right)$	$\left([0.57,0.76]e^{i2\pi[0.63,0.65]}, [0.48,0.68]e^{i2\pi[0.54,0.67]}\right)$	$\left([0.59,0.72]e^{i2\pi[0.54,0.84]}, [0.45,0.74]e^{i2\pi[0.45,0.53]}\right)$	$\left([0.58,0.69]e^{i2\pi[0.54,0.70]}, [0.55,0.62]e^{i2\pi[0.45,0.58]}\right)$

Step 2: In Step 2, the data from Tables 1–4 is integrated into a single matrix using the proposed CIVFFWA and CIVFFWG operators, applying the weighted vector ω = 0.4, 0.2, 0.1, 0.3. These weights indicate the significance of each expert's input in the decision-making process. The consolidated matrices, shown in Tables 5 and 6, offer a comprehensive synthesis of expert evaluations, ensuring an accurate representation of both individual and collective judgments.

Step 3 (i) – Again using the CIVFFWA operator, where $\varphi = (0.3,0.3,0.2,0.2)$ we get the preference values as:

$$\xi_1 = \left([0.55,0.81]e^{i[0.64,0.76]}, [0.54,0.58]e^{i[0.53,0.67]}\right)$$
$$\xi_2 = \left([0.67,0.70]e^{i[0.47,0.84]}, [0.49,0.57]e^{i[0.63,0.65]}\right)$$
$$\xi_3 = \left([0.63,0.72]e^{i[0.63,0.83]}, [0.47,0.64]e^{i[0.47,0.56]}\right)$$
$$\xi_4 = \left([0.46,0.68]e^{i[0.71,0.75]}, [0.42,0.65]e^{i[0.46,0.58]}\right)$$

Step 3 (ii) – Again using the CIVFFWG operator, we get the preference values as:

$$\xi_1 = \left([0.56,0.82]e^{i[0.65,0.77]}, [0.51,0.68]e^{i[0.52,0.64]}\right)$$
$$\xi_2 = \left([0.64,0.73]e^{i[0.49,0.84]}, [0.62,0.67]e^{i[0.47,0.59]}\right)$$
$$\xi_3 = \left([0.63,0.77]e^{i[0.47,0.68]}, [0.42,0.59]e^{i[0.44,0.62]}\right)$$
$$\xi_4 = \left([0.58,0.74]e^{i[0.64,0.73]}, [0.46,0.59]e^{i[0.43,0.47]}\right)$$

Step 4 (i) – Calculating the score values of all preference values of Table 5 using Definition 7.

$$score(\xi_1) = 0.32, score(\xi_2) = 0.31, score(\xi_3) = 0.38, score(\xi_4) = 0.33$$

Step 4 (ii) – Again computing the score values of all preference values of Table 6.

$$score(\xi_1) = 0.35, score(\xi_2) = 0.27, score(\xi_3) = 0.39, score(\xi_4) = 0.37$$

Step 5 (i) – Ranking of all alternative is $\xi_3 \succ \xi_4 \succ \xi_1 \succ \xi_2$.

Step 5 (ii) – Ranking of all alternative is $\xi_3 \succ \xi_4 \succ \xi_1 \succ \xi_2$.

Step 6 – Thus the best option is "*Project-Based Learning*".

Table 5
By CIVFFWA Approach

	C_1	C_2	C_3	C_4
A_1	$\left([0.54,0.73]e^{i2\pi[0.67,0.83]}, [0.52,0.65]e^{i2\pi[0.64,0.72]}\right)$	$\left([0.64,0.78]e^{i2\pi[0.53,0.76]}, [0.48,0.64]e^{i2\pi[0.53,0.69]}\right)$	$\left([0.59,0.67]e^{i2\pi[0.55,0.67]}, [0.62,0.72]e^{i2\pi[0.61,0.68]}\right)$	$\left([0.48,0.72]e^{i2\pi[0.44,0.73]}, [0.55,0.76]e^{i2\pi[0.53,0.64]}\right)$
A_2	$\left([0.54,0.65]e^{i2\pi[0.57,0.76]}, [0.56,0.68]e^{i2\pi[0.64,0.79]}\right)$	$\left([0.55,0.65]e^{i2\pi[0.55,0.73]}, [0.62,0.73]e^{i2\pi[0.53,0.64]}\right)$	$\left([0.46,0.57]e^{i2\pi[0.43,0.52]}, [0.52,0.73]e^{i2\pi[0.64,0.78]}\right)$	$\left([0.51,0.56]e^{i2\pi[0.45,0.67]}, [0.62,0.76]e^{i2\pi[0.47,0.87]}\right)$
A_3	$\left([0.63,0.75]e^{i2\pi[0.56,0.65]}, [0.54,0.66]e^{i2\pi[0.65,0.74]}\right)$	$\left([0.52,0.71]e^{i2\pi[0.64,0.81]}, [0.53,0.64]e^{i2\pi[0.64,0.70]}\right)$	$\left([0.59,0.72]e^{i2\pi[0.54,0.84]}, [0.45,0.74]e^{i2\pi[0.45,0.53]}\right)$	$\left([0.62,0.66]e^{i2\pi[0.56,0.75]}, [0.54,0.67]e^{i2\pi[0.64,0.77]}\right)$
A_4	$\left([0.62,0.74]e^{i2\pi[0.65,0.73]}, [0.46,0.76]e^{i2\pi[0.55,0.76]}\right)$	$\left([0.45,0.64]e^{i2\pi[0.61,0.79]}, [0.54,0.59]e^{i2\pi[0.64,0.72]}\right)$	$\left([0.58,0.69]e^{i2\pi[0.55,0.74]}, [0.62,0.73]e^{i2\pi[0.53,0.66]}\right)$	$\left([0.65,0.68]e^{i2\pi[0.53,0.72]}, [0.52,0.60]e^{i2\pi[0.63,0.68]}\right)$

Table 6
By CIVFFWG Approach

	C_1	C_2	C_3	C_4
A_1	$\left([0.56,0.71]e^{i2\pi[0.62,0.80]}, [0.53,0.64]e^{i2\pi[0.64,0.71]}\right)$	$\left([0.58,0.69]e^{i2\pi[0.55,0.74]}, [0.62,0.73]e^{i2\pi[0.53,0.66]}\right)$	$\left([0.59,0.67]e^{i2\pi[0.55,0.67]}, [0.62,0.72]e^{i2\pi[0.61,0.68]}\right)$	$\left([0.58,0.74]e^{i2\pi[0.54,0.75]}, [0.57,0.76]e^{i2\pi[0.53,0.64]}\right)$
A_2	$\left([0.52,0.66]e^{i2\pi[0.57,0.76]}, [0.55,0.67]e^{i2\pi[0.65,0.79]}\right)$	$\left([0.55,0.65]e^{i2\pi[0.55,0.73]}, [0.62,0.73]e^{i2\pi[0.53,0.64]}\right)$	$\left([0.55,0.77]e^{i2\pi[0.65,0.78]}, [0.54,0.64]e^{i2\pi[0.64,0.72]}\right)$	$\left([0.68,0.78]e^{i2\pi[0.65,0.73]}, [0.61,0.76]e^{i2\pi[0.53,0.78]}\right)$
A_3	$\left([0.68,0.79]e^{i2\pi[0.56,0.65]}, [0.52,0.62]e^{i2\pi[0.63,0.77]}\right)$	$\left([0.46,0.57]e^{i2\pi[0.43,0.52]}, [0.52,0.73]e^{i2\pi[0.64,0.78]}\right)$	$\left([0.59,0.72]e^{i2\pi[0.54,0.84]}, [0.45,0.74]e^{i2\pi[0.45,0.53]}\right)$	$\left([0.65,0.68]e^{i2\pi[0.53,0.72]}, [0.52,0.60]e^{i2\pi[0.63,0.68]}\right)$
A_4	$\left([0.66,0.79]e^{i2\pi[0.65,0.73]}, [0.52,0.76]e^{i2\pi[0.53,0.78]}\right)$	$\left([0.45,0.64]e^{i2\pi[0.61,0.79]}, [0.54,0.59]e^{i2\pi[0.64,0.72]}\right)$	$\left([0.63,0.78]e^{i2\pi[0.56,0.76]}, [0.58,0.64]e^{i2\pi[0.53,0.60]}\right)$	$\left([0.62,0.66]e^{i2\pi[0.56,0.75]}, [0.54,0.67]e^{i2\pi[0.64,0.77]}\right)$

Tables 7 and 8 present the score functions for all methods, offering a detailed performance comparison.

Table 7
Scores Values of All Proposed Methods

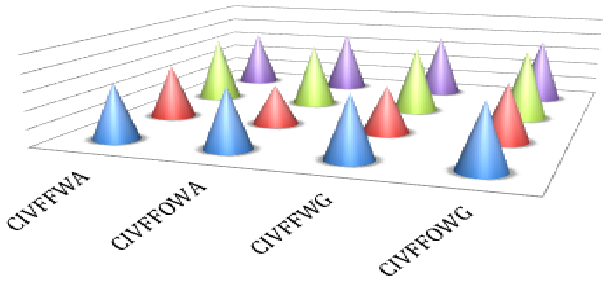
Options	CIVFFWA	CIVFFOWA	CIVFFWG	CIVFFOWG
Lecture-Based Teaching	0.32	0.34	0.35	0.36
Collaborative Learning Method	0.31	0.23	0.27	0.33
Project-Based Learning	0.38	0.37	0.39	0.41
Case-Based Learning	0.33	0.36	0.37	0.38

Table 8
Ranking of Alternatives Based on Score Values

Methods	Score Values	Ranking of Alternatives
CIVFFWA	$score(A_3) \succ score(A_4) \succ score(A_1) \succ score(A_2)$	$A_3 \succ A_4 \succ A_1 \succ A_2$
CIVFFOWA	$score(A_3) \succ score(A_4) \succ score(A_1) \succ score(A_2)$	$A_3 \succ A_4 \succ A_1 \succ A_2$

CIVFFWG	$score(A_3) \succ score(A_4) \succ score(A_1) \succ score(A_2)$	$A_3 \succ A_4 \succ A_1 \succ A_2$
CIVFFOWG	$score(A_3) \succ score(A_4) \succ score(A_1) \succ score(A_2)$	$A_3 \succ A_4 \succ A_1 \succ A_2$

Additionally, Figures 2 and 3 visually illustrate the ranking and range of each method, highlighting their relative effectiveness. Together, these figures provide a comprehensive overview of the strengths and limitations of the evaluated methods.



	CIVFFWA	CIVFFOWA	CIVFFWG	CIVFFOWG
LBT	0.32	0.34	0.35	0.36
CLM	0.31	0.23	0.27	0.33
PBL	0.38	0.37	0.39	0.41
CBL	0.33	0.36	0.37	0.38

Fig.2. Ranking of the Proposed Methods

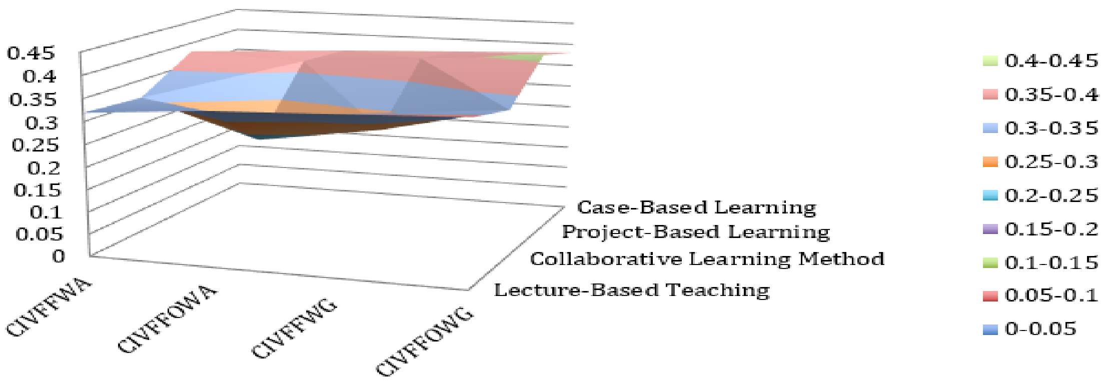


Fig.3. Graphical Presentation of All Methods

7. Comparative Analysis

To validate our research, we compare our model and methods with existing ones using model-wise and methods-wise approaches. This evaluation highlights the strengths and effectiveness of our proposed approach.

Model-Wise: CIVFSs improve traditional fuzzy models by using complex numbers to handle higher-dimensional hesitation and uncertainty. They work well in areas like pattern recognition, control systems, and decision-making. Unlike IVIF-Set, IVPyF-Set, and IVFF-Set, which use real numbers, CIVFSs offer a more flexible and precise way to model complex systems. They can manage uncertainty better than other models, making them useful for a wide range of applications $(n + g \leq 1, n^2 + g^2 \leq 1, n^3 + g^3 \leq 1)$. CIVFSs provide a more comprehensive model. While real-number-based models structure hesitation and uncertainty well, they cannot handle complex numbers, limiting their use for intricate data. Despite this, they remain useful for managing uncertainty and supporting decision-making in various fields. Meanwhile, CIVIFS and CIVPyFS with limitations on magnitude and phase

$\left(n+g\le 1,\left(\frac{q}{2\pi}\right)+\left(\frac{s}{2\pi}\right)\le 1, n^2+g^2\le 1,\left(\frac{q}{2\pi}\right)^2+\left(\frac{s}{2\pi}\right)^2\le 1\right)$ respectively were improving vagueness representation. By incorporating complex numbers, the proposed model enhances decision analysis, providing a robust framework for handling intricate and uncertain data. It improves uncertainty management, especially in complex scenarios, leading to more precise decision-making. But CIVFF- Set introduces complex-valued membership and non-membership degrees with $n^3+g^3\le 1$ and$\left(\frac{q}{2\pi}\right)^3+\left(\frac{s}{2\pi}\right)^3\le 1$. For example, the value $Ћ=\left([0.62,0.73]e^{i[0.71,0.75]},[0.68,0.84]e^{i[0.64,0.82]}\right)$ fails satisfy the condition for these sets which are: $n+g\le 1$, $\left(\frac{q}{2\pi}\right)+\left(\frac{s}{2\pi}\right)\le 1$ and $n^2+g^2\le 1$. $\left(\frac{q}{2\pi}\right)^2+\left(\frac{s}{2\pi}\right)^2\le 1$. i.e., $0.73+0.84=1.57\succ 1$ and $\left(\frac{0.75}{2\pi}\right)+\left(\frac{0.82}{2\pi}\right)=1.57\succ 1$. This data cannot be handled using CIVIFS. Next $(0.73)^2+(0.84)^2=1.23\succ 1$ and $\left(\frac{0.75}{2\pi}\right)^2+\left(\frac{0.82}{2\pi}\right)^2=1.23\succ 1$. Therefore, CIVPyFS is unable to process this information. Now $(0.73)^3+(0.84)^3=0.98\prec 1$ and $\left(\frac{0.75}{2\pi}\right)^3+\left(\frac{0.82}{2\pi}\right)^3=0.97\prec 1$. Thus, CIVFFS can effectively process this information, as shown in Table 9.

Table 9
Comparison of New Model with Existing Models

Models	Uncertainty	Falsity	Indeterminacy	Periodicity	Multi-Dimensional Information	Power in Square	Power in Cube
FSs[1]	1	0	0	0	0	0	0
IFSs[2]	1	1	1	0	0	0	0
IVIFSs[8]	1	1	1	0	0	0	0
PyFSs[9]	1	1	1	0	0	1	0
IVPyFSs[12]	1	1	1	0	0	1	0
FFSs[15]	1	1	1	0	0	0	1
IVFFSs[17]	1	1	1	0	0	0	1
CFSs[18]	1	0	0	1	1	0	0
CIFSs[19]	1	1	1	1	1	0	0
CIVIFSs[23]	1	1	1	1	1	0	0
CPyFSs[25]	1	1	1	1	1	1	0
CIVPyFSs[30]	1	1	1	1	1	1	0
CIVFFS	1	1	1	1	1	1	1

This framework represents a significant advancement over existing models, offering greater adaptability in complex decision-making. Its flexible structure allows it to accommodate diverse contexts, ensuring more accurate and reliable outcomes. Unlike traditional models, it efficiently handles uncertainty and hesitation, enabling more precise and informed decision-making across various applications.

Methods-Wise: To validate our methods, we will compare them with existing approaches to demonstrate their flexibility and reliability. Benchmarking against established models will highlight their superior adaptability and performance. Rigorous testing will confirm their effectiveness in handling diverse scenarios, proving their efficiency and real-world applicability.

8. Conclusion and Implications

This research introduced CIVFFS, a novel framework integrating CFFS and IVFFS to enhance decision-making by representing two-dimensional information. By allowing both membership and non-membership functions to take interval values, CIVFFS improves the modelling of hesitation, uncertainty, and complexity. We developed new approaches—CIVFFWA, CIVFFOWA, CIVFFWG, and CIVFFOWG—and analysed their fundamental properties. To demonstrate practical applicability, we applied these methods to evaluate classroom teaching modes, assessing key influencing factors and ranking them based on score values. The findings confirmed the effectiveness of the proposed methods in handling time-sensitive decision-making problems. This study advances decision-making methodologies by offering a robust framework for managing complexity and uncertainty, particularly in education and other practical domains, ensuring greater accuracy and reliability than existing models.

Author Contributions

Conceptualization, G.W.; methodology, Z.W.; software, Z.W.; validation, Z.W.; formal analysis, G.W.; investigation, G.W; resources, G.W.; data curation, Z.W.; writing—original draft preparation, Z.W.; writing—review and editing, G.W.; visualization, G.W.; supervision, G.W.; project administration, G.W.; funding acquisition, G.W. All authors have read and agreed to the published version of the manuscript.

Data Availability Statement

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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