Decision Making: Applications in Management and Engineering
Vol. 5, Issue 1, 2022, pp. 27-49.
ISSN: 2560-6018
eISSN: 2620-0104
DOI: https://doi.org/10.31181/dmame0316022022r

PARETO-EFFICIENT STRATEGIES IN 2-PERSON GAMES IN STAIRCASE-FUNCTION CONTINUOUS AND FINITE SPACES

Vadim Romanuke [1*]

[1]Polish Naval Academy, Poland

Received: 16 September 2021;
Accepted: 25 December 2021;
Available online: 16 February 2022.

Original scientific paper

Abstract: *A tractable method of solving noncooperative 2-person games in which strategies are staircase functions is suggested. The solution is meant to be Pareto-efficient. The method considers any 2-person staircase-function game as a succession of 2-person games in which strategies are constants. For a finite staircase-function game, each constant-strategy game is a bimatrix game whose size is sufficiently small to solve it in a reasonable time. It is proved that any staircase-function game has a single Pareto-efficient situation if every constant-strategy game has a single Pareto-efficient situation, and vice versa. Besides, it is proved that, whichever the staircase-function game continuity is, any Pareto-efficient situation of staircase function-strategies is a stack of successive Pareto-efficient situations in the constant-strategy games. If a staircase-function game has two or more Pareto-efficient situations, the best efficient situation is found by holding it the farthest from the pair of the most unprofitable payoffs.*

Key words: *Game theory, payoff functional, Pareto efficiency, staircase-function strategy, bimatrix game.*

1. Introduction

A struggle for optimizing the distribution of limited resources between two sides is mathematically formalized and modeled as a noncooperative 2-person game (Vorob'yov, 1984; Vorob'yov, 1985; Myerson, 1997; Osborne, 2003). Qualitative properties of the 2-person game strongly depend on the sets of the persons' (players') pure strategies. The properties including payoff attraction are far simpler for the case of when the sets are countable. The simplest 2-person game is when the sets are finite. In this case, the game is called bimatrix (Vorob'yov, 1958; Moulin, 1981; Harsanyi & Selten, 1988).

The bimatrix game is nonetheless an intricate model of an interaction process. Although the interaction process itself is a selection-and-payoff event (or a series of such events), not involving any differentiation or integration, the interpretation of the

* Corresponding author.
E-mail address: v.romanuke@amw.gdynia.pl (V. Romanuke)

eventual result appears difficult enough. First, the optimality or the best decision (solution) has multiple types (Vorob'yov, 1985; Harsanyi & Selten, 1988; Belhaiza et al., 2012). This is so because the optimality requires equilibrium, efficiency (profitability), and fairness. These types are often contradictory in a 2-person game. For instance, an equilibrium situation may be efficient for a player while it is not profitable for the other player (thus the respective payoffs are unfair) (Kayı & Ramaekers, 2010; Alva & Manjunath, 2020). Second, a bimatrix game may have multiple equilibria along with multiple Pareto-efficient situations (Vorob'yov, 1985; Romanuke, 2018b; Fu, 2021). This induces the solution uncertainty even if there are simultaneously efficient and fair equilibria (Kayı & Ramaekers, 2010; Ke et al., 2021). Furthermore, even a bimatrix game may have a continuum of equilibria, wherein the best decision selection is far more difficult (Vorob'yov, 1984; Moulin, 1981; Harsanyi & Selten, 1988).

Infinite and, moreover, continuous 2-person games are far more complicated than bimatrix (i. e., finite) games. Whereas the bimatrix game has at least an equilibrium (generally speaking, in mixed strategies), an infinite game may not have an equilibrium or it just is indeterminable (Vorob'yov, 1984; Kontogiannis et al., 2009; Osborne, 2003). The best option, therefore, is to deal with a finite 2-person game trivially rendered to a bimatrix game (Schelling, 1980; Moulin, 1981; Vorob'yov, 1984; Romanuke & Kamburg, 2016; Romanuke, 2020).

A complication arises as the structure of the player's pure strategy becomes more complex. The most trivial strategy is a decision corresponding to a one-stage event whose duration through time is (negligibly) short. However, a strategy can be a multi-stage process like a staircase-function (of time) defined on a time interval (Kayı & Ramaekers, 2010; Zheng et al, 2019; Kim et al, 2019; Li et al, 2020; Ke et al., 2021). In a pure strategy situation, a pair of such staircase-function strategies (from both the players) is mapped into a real value. When each of the players possesses a finite set of such function-strategies, the staircase-function game is easily rendered down to a bimatrix game and (tried to be) solved, whichever the solution is (Leyton-Brown & Shoham, 2008; Romanuke & Kamburg, 2016; Romanuke, 2020). Such rendering is impossible if the set of the player's function-strategies is either infinite or continuous.

2. Motivation

If the strategy is a time function defined on a closed (time) interval, it is made staircase by imposing a natural constraint on the elementary action of a player. In real-world practice, the continuity of a process is an ill-posed assumption, so any process through a definite time interval is a finite set of elementary actions. Thus, a staircase function during an elementary action can be considered constant. So the staircase-function game is formed naturally, where the natural constraint is made of itself by the laws of reality (Schelling, 1980; Vorob'yov, 1985; Kayı & Ramaekers, 2010; Romanuke, 2020; Ke et al., 2021; Fu, 2021).

To make a staircase-function game finite, the set of possible values of the player's pure strategy should be finite. In such a staircase-function bimatrix game the player's selection of a pure strategy means using a staircase function on a time interval whereon every pure strategy is defined. The total number of the player's pure strategies in the staircase-function bimatrix game is determined by the number of "stair" subintervals and the number of possible values of the player's pure strategy (staircase function). For example, if the number of subintervals is 6, and the number of possible values of the player's pure strategy is just 5, then there are $5^6 = 15625$

possible pure strategies at this player, where every strategy is a 6-subinterval 5-staircased function of time. The respective bimatrix 15625×15625 game even in this trivialized case appears to be big enough. In a more real example, when every strategy, say, is an 8-subinterval 10-staircased function of time, the respective bimatrix $10^8 \times 10^8$ game appears to be intractably gigantic: a solution cannot be found in a reasonable amount of time (and on a reasonably expensive hardware) among 10 quadrillion situations in such a game. This means that straightforwardly solving staircase-function bimatrix games (i. e., 2-person games in staircase-function finite spaces) is impracticable.

Another question is what the solution should be. Although the property of solution stability is considered important, the equilibrium in 2-person games often is unprofitable for one of the players. For example, in a 2×2 game with payoff matrices

$$\mathbf{K} = \begin{bmatrix} 5 & 3 \\ 4.8 & 6 \end{bmatrix} \tag{1}$$

and

$$\mathbf{H} = \begin{bmatrix} 2 & 2 \\ 9 & 1 \end{bmatrix} \tag{2}$$

of the first and second players, respectively, there is a single equilibrium with payoffs $\{5, 2\}$. By the way, this equilibrium is Pareto-efficient. Besides, there are another two efficient situations with payoffs $\{4.8, 9\}$ and $\{6, 1\}$. Obviously, the first player must realize that the second player will definitely stick to one's first strategy (which, apart form being equilibrium strategy, nonstrictly dominates the second strategy). However, the first player's decision to hold to the first strategy would be quite unprofitable for the second player. At the same time, the first player loss is only 4 % if to select one's second strategy (and receive 4.8 instead of 5), whereas the second player receives 9 (instead of 2). The reasoning does not change if the second player's payoff is any amount above 9 (strictly speaking, the respective situation is efficient if only the amount is greater than 2) in matrix (2). This example shows that an equilibrium called to keep the property of solution stability may be unstable under certain circumstances. Therefore, Pareto-efficient strategies in 2-person games are first to be checked. Although they are not formally stable, their stability will likely be induced in the way described above. The formal stability of equilibrium is likely to be shattered contrariwise.

3. Objective and tasks to be fulfilled

Issuing from the impracticability of straightforwardly solving finite noncooperative 2-person games in staircase-function finite spaces, the objective is to develop a tractable method of solving such games. The solution is meant to be Pareto-efficient. For meeting the objective, the following six tasks are to be fulfilled:

1. To formalize a noncooperative 2-person game, in which the players' strategies are staircase functions of time, whereas the time is discrete. In such a game, the set of the player's pure strategies is a continuum of staircase functions.

2. To consider the property of Pareto efficiency in a staircase-function game.

3. To suggest a method of solving finite 2-person staircase-function games (i. e., games in staircase-function finite spaces) by using the Pareto-efficiency criterion.

4. To give an example of how the suggested method is applied.
5. To discuss practical applicability and scientific significance of the method.
6. To make an appropriate conclusion on it. An outlook for furthering the study should be made as well.

4. A 2-person game defined on a product of functional spaces

In a noncooperative 2-person game, in which the player's pure strategy is a function, let each of the players use strategies defined almost everywhere on (time) interval $[t_1; t_2]$ by $t_2 > t_1$. Denote a strategy of the first player by $x(t)$ and a strategy of the second player by $y(t)$. These functions are presumed to be bounded, i. e.

$$a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max} \tag{3}$$

and

$$b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}, \tag{4}$$

defined almost everywhere on $[t_1; t_2]$. Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space of functions of time:

$$X = \left\{ x(t),\, t \in [t_1; t_2],\, t_1 < t_2 : a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max} \right\} \subset \mathbb{L}_2[t_1; t_2] \tag{5}$$

and

$$Y = \left\{ y(t),\, t \in [t_1; t_2],\, t_1 < t_2 : b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max} \right\} \subset \mathbb{L}_2[t_1; t_2] \tag{6}$$

are the sets of the players' pure strategies.

The first player's payoff in situation

$$\{x(t),\, y(t)\} \tag{7}$$

is $K(x(t), y(t))$ presumed to be an integral functional (Edwards, 1965; Romanuke, 2020):

$$K(x(t), y(t)) = \int\limits_{[t_1; t_2]} f(x(t), y(t), t)\, d\mu(t), \tag{8}$$

where

$$f(x(t), y(t), t) \tag{9}$$

is a function of $x(t)$ and $y(t)$ explicitly including t. The second player's payoff in situation (7) is $H(x(t), y(t))$ presumed to be an integral functional also:

$$H(x(t), y(t)) = \int\limits_{[t_1; t_2]} g(x(t), y(t), t)\, d\mu(t), \tag{10}$$

where

$$g\left(x(t),\, y(t),\, t\right) \tag{11}$$

is a function of $x(t)$ and $y(t)$ explicitly including t just like (9). Hence, the continuous 2-person game

$$\left\langle \left\{X,\, Y\right\},\, \left\{K\left(x(t),\, y(t)\right),\, H\left(x(t),\, y(t)\right)\right\}\right\rangle \tag{12}$$

is defined on product

$$X \times Y \subset \mathbb{L}_2\left[t_1;\, t_2\right] \times \mathbb{L}_2\left[t_1;\, t_2\right] \tag{13}$$

of rectangular functional spaces (5) and (6) of players' pure strategies.

5. A 2-person staircase-function game

As it has been above-mentioned, the staircase-function game is formed naturally, so denote by N the number of the elementary actions could be made by a player, where obviously $N \in \mathbb{N} \setminus \{1\}$. In fact, it is the number of "stair" subintervals at which the player's pure strategy is constant. Then the player's pure strategy is a staircase function which may have up to N different values (but no more than that).

If $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$ are time points at which the staircase-function strategy changes or may change its value, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2\,, \tag{14}$$

then

$$x_i = x\left(\tau^{(i)}\right) \text{ and } y_i = y\left(\tau^{(i)}\right) \text{ by } i = \overline{0,\, N} \tag{15}$$

are the values of pure strategies of the first and second players, respectively. The staircase-function strategies are right-continuous (Edwards, 1965):

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)} + \varepsilon\right) = x\left(\tau^{(i)}\right) \text{ and } \lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)} + \varepsilon\right) = y\left(\tau^{(i)}\right) \text{ for } i = \overline{1,\, N-1} \tag{16}$$

by

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)} - \varepsilon\right) \neq x\left(\tau^{(i)}\right) \text{ and } \lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)} - \varepsilon\right) \neq y\left(\tau^{(i)}\right) \text{ for } i = \overline{1,\, N-1}\,. \tag{17}$$

As an exception,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(N)} - \varepsilon\right) = x\left(\tau^{(N)}\right) \text{ and } \lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(N)} - \varepsilon\right) = y\left(\tau^{(N)}\right), \tag{18}$$

so $x_{N-1} = x_N$ and $y_{N-1} = y_N$. Then constant values (15) by (14) mean that game (12) is a 2-person staircase-function game. The staircase-function game can be thought of as it is a succession of N continuous games

$$\left\langle \left\{ [a_{\min}; a_{\max}], [b_{\min}; b_{\max}] \right\}, \left\{ K(\alpha_i, \beta_i), H(\alpha_i, \beta_i) \right\} \right\rangle \tag{19}$$

each defined on rectangle

$$[a_{\min}; a_{\max}] \times [b_{\min}; b_{\max}] \tag{20}$$

by

$$\alpha_i = x(t) \in [a_{\min}; a_{\max}] \text{ and } \beta_i = y(t) \in [b_{\min}; b_{\max}]$$
$$\forall t \in \left[\tau^{(i-1)}; \tau^{(i)}\right) \text{ for } i = \overline{1, N-1} \text{ and } \forall t \in \left[\tau^{(N-1)}; \tau^{(N)}\right], \tag{21}$$

where the factual first player's payoff in situation $\{\alpha_i, \beta_i\}$ is

$$K(\alpha_i, \beta_i) = \int\limits_{\left[\tau^{(i-1)}; \tau^{(i)}\right)} f(\alpha_i, \beta_i, t)\, d\mu(t) \quad \forall i = \overline{1, N-1} \tag{22}$$

and

$$K(\alpha_N, \beta_N) = \int\limits_{\left[\tau^{(N-1)}; \tau^{(N)}\right]} f(\alpha_N, \beta_N, t)\, d\mu(t), \tag{23}$$

and the factual second player's payoff in situation $\{\alpha_i, \beta_i\}$ is

$$H(\alpha_i, \beta_i) = \int\limits_{\left[\tau^{(i-1)}; \tau^{(i)}\right)} g(\alpha_i, \beta_i, t)\, d\mu(t) \quad \forall i = \overline{1, N-1} \tag{24}$$

and

$$H(\alpha_N, \beta_N) = \int\limits_{\left[\tau^{(N-1)}; \tau^{(N)}\right]} g(\alpha_N, \beta_N, t)\, d\mu(t). \tag{25}$$

The payoff in situation $\{\alpha_i, \beta_i\}$ can be thought of as it is the payoff on a "stair" subinterval i, which is $\left[\tau^{(i-1)}; \tau^{(i)}\right)$ for $\forall i = \overline{1, N-1}$ and $\left[\tau^{(N-1)}; \tau^{(N)}\right]$ (when $i = N$). A pure-strategy situation in the staircase-function game (12) is a succession of N situations $\left\{\{\alpha_i, \beta_i\}\right\}_{i=1}^{N}$ in games (19).

Theorem 1. In a pure-strategy situation of the staircase-function game (12), represented as a succession of N continuous games (19), functionals (8) and (10) are re-written as subinterval-wise sums.

$$K(x(t), y(t)) = \sum_{i=1}^{N} K(\alpha_i, \beta_i) =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)}; \tau^{(i)}\right)} f(\alpha_i, \beta_i, t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)}; \tau^{(N)}\right]} f(\alpha_N, \beta_N, t)\, d\mu(t) \tag{26}$$

and

$$H\left(x(t),\, y(t)\right)=\sum_{i=1}^{N} H\left(\alpha_i,\, \beta_i\right)=$$
$$=\sum_{i=1}^{N-1}\int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(\alpha_i,\, \beta_i,\, t\right)d\mu(t)+\int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(\alpha_N,\, \beta_N,\, t\right)d\mu(t), \tag{27}$$

respectively.

Proof. Each of functions (9) and (11) in situation $\left\{\alpha_i,\, \beta_i\right\}$ by (19) is some function of time t. Thus, denote a function corresponding to (9) by $\psi_i(t)$. For situation $\left\{\alpha_i,\, \beta_i\right\}$ on half-subinterval $\left[\tau^{(i-1)};\, \tau^{(i)}\right)$ by $i=\overline{1,\, N-1}$ function

$$\psi_i(t)=0\ \ \forall t\notin\left[\tau^{(i-1)};\, \tau^{(i)}\right), \tag{28}$$

and for situation $\left\{\alpha_N,\, \beta_N\right\}$ on subinterval $\left[\tau^{(N-1)};\, \tau^{(N)}\right]$ function

$$\psi_N(t)=0\ \ \forall t\notin\left[\tau^{(N-1)};\, \tau^{(N)}\right]. \tag{29}$$

Therefore,

$$f\left(x(t),\, y(t),\, t\right)=\sum_{i=1}^{N}\psi_i(t) \tag{30}$$

in a pure-strategy situation $\left\{x(t),\, y(t)\right\}$ of the staircase-function game (9), by using (29) and (30). Consequently,

$$K\left(x(t),\, y(t)\right)=\int\limits_{[t_1;\, t_2]} f\left(x(t),\, y(t),\, t\right)d\mu(t)=$$
$$=\sum_{i=1}^{N-1}\int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)}\psi_i(t)d\mu(t)+\int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]}\psi_N(t)d\mu(t)=$$
$$=\sum_{i=1}^{N-1}\int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(\alpha_i,\, \beta_i,\, t\right)d\mu(t)+\int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(\alpha_N,\, \beta_N,\, t\right)d\mu(t)=$$
$$=\sum_{i=1}^{N} K\left(\alpha_i,\, \beta_i\right) \tag{31}$$

in a pure-strategy situation $\left\{x(t),\, y(t)\right\}$ of the staircase-function game (9). Obviously, subinterval-wise sum (27) is proved similarly to (28) — (31). $\square$

It is clear that Theorem 1, although not providing a method of solving the staircase-function game, simplifies it. In payoff terms, Theorem 1 allows considering each game (19) separately. The stack of successive situations $\left\{\left\{\alpha_i,\, \beta_i\right\}\right\}_{i=1}^{N}$ is a (staircase) situation in the respective 2-person staircase-function game (12).

6. When a Pareto-efficient stack is single

The occurrence when every subinterval 2-person game has a single Pareto-efficient situation is rare. The likelihood of such an occurrence even for finite staircase-function games is roughly less than 1 %. Nevertheless, there is an interesting assertion addressed to this case.

Theorem 2. If each of N games (19) by (14) — (18) and (20) — (25) has a single Pareto-efficient situation, then the respective 2-person staircase-function game (12) has a single Pareto-efficient situation, which is the stack of successive Pareto-efficient situations in games (19).

Proof. Let $\left\{\alpha_i^*, \beta_i^*\right\}$ be the single efficient situation in the game on "stair" subinterval i. This implies that both a pair of inequalities

$$K\left(\alpha_i, \beta_i\right) \geqslant K\left(\alpha_i^*, \beta_i^*\right) \tag{32}$$

and

$$H\left(\alpha_i, \beta_i\right) > H\left(\alpha_i^*, \beta_i^*\right) \tag{33}$$

and a pair of inequalities

$$K\left(\alpha_i, \beta_i\right) > K\left(\alpha_i^*, \beta_i^*\right) \tag{34}$$

and

$$H\left(\alpha_i, \beta_i\right) \geqslant H\left(\alpha_i^*, \beta_i^*\right) \tag{35}$$

are impossible for any $\alpha_i \in \left[a_{\min}; a_{\max}\right]$ and $\beta_i \in \left[b_{\min}; b_{\max}\right]$. In other words, inequalities (32) and (33) are simultaneously impossible, and inequalities (34) and (35) are simultaneously impossible as well. Suppose that $\exists\, \alpha_i^{(0)} \in \left[a_{\min}; a_{\max}\right]$ and $\exists\, \beta_i^{(0)} \in \left[b_{\min}; b_{\max}\right]$ such that

$$K\left(\alpha_i^{(0)}, \beta_i^{(0)}\right) > K\left(\alpha_i^*, \beta_i^*\right). \tag{36}$$

Inequality (36) implies that inequality

$$H\left(\alpha_i^{(0)}, \beta_i^{(0)}\right) < H\left(\alpha_i^*, \beta_i^*\right) \tag{37}$$

must hold because otherwise situation $\left\{\alpha_i^*, \beta_i^*\right\}$ will not be efficient. However, inequalities (36) and (37) imply that situation $\left\{\alpha_i^{(0)}, \beta_i^{(0)}\right\}$ is efficient, which is impossible due to $\left\{\alpha_i^*, \beta_i^*\right\}$ is the single efficient situation. Therefore, inequality (34) is impossible, and impossibility of the remaining inequalities (32), (33), (35) is proved similarly.

As inequalities (32) — (35) are impossible for every $i = \overline{1, N}$ then each of the inequalities

$$\sum_{i=1}^{N} K(\alpha_i, \beta_i) \geqslant \sum_{i=1}^{N} K(\alpha_i^*, \beta_i^*), \quad (38)$$

$$\sum_{i=1}^{N} H(\alpha_i, \beta_i) > \sum_{i=1}^{N} H(\alpha_i^*, \beta_i^*), \quad (39)$$

$$\sum_{i=1}^{N} K(\alpha_i, \beta_i) > \sum_{i=1}^{N} K(\alpha_i^*, \beta_i^*), \quad (40)$$

$$\sum_{i=1}^{N} H(\alpha_i, \beta_i) \geqslant \sum_{i=1}^{N} H(\alpha_i^*, \beta_i^*) \quad (41)$$

for any $\alpha_i \in [a_{\min}; a_{\max}]$ and $\beta_i \in [b_{\min}; b_{\max}]$ is impossible as well. By the efficiency definition, owing to Theorem 1, this implies that stack $\left\{\left\{\alpha_i^*, \beta_i^*\right\}\right\}_{i=1}^{N}$ is a Pareto-efficient situation in the respective 2-person staircase-function game (12).

Suppose that there is another stack which is also Pareto-efficient. Consider the case when $N = 2$. First, let stack

$$\left\{\left\{\alpha_1^{(0)}, \beta_1^*\right\}, \left\{\alpha_2^*, \beta_2^*\right\}\right\} \quad (42)$$

be a Pareto-efficient situation by $\alpha_1^{(0)} \neq \alpha_1^*$. This implies that both a pair of inequalities

$$K(\alpha_1, \beta_1) + K(\alpha_2, \beta_2) \geqslant K(\alpha_1^{(0)}, \beta_1^*) + K(\alpha_2^*, \beta_2^*) \quad (43)$$

and

$$H(\alpha_1, \beta_1) + H(\alpha_2, \beta_2) > H(\alpha_1^{(0)}, \beta_1^*) + H(\alpha_2^*, \beta_2^*) \quad (44)$$

and a pair of inequalities

$$K(\alpha_1, \beta_1) + K(\alpha_2, \beta_2) > K(\alpha_1^{(0)}, \beta_1^*) + K(\alpha_2^*, \beta_2^*) \quad (45)$$

and

$$H(\alpha_1, \beta_1) + H(\alpha_2, \beta_2) \geqslant H(\alpha_1^{(0)}, \beta_1^*) + H(\alpha_2^*, \beta_2^*) \quad (46)$$

are impossible for any $\alpha_i \in [a_{\min}; a_{\max}]$ and $\beta_i \in [b_{\min}; b_{\max}]$ by $i = 1$ and $i = 2$. Plugging $\alpha_2 = \alpha_2^*$ and $\beta_2 = \beta_2^*$ in the left sides of inequalities (43) — (46) gives a pair of inequalities

$$K(\alpha_1, \beta_1) \geqslant K(\alpha_1^{(0)}, \beta_1^*), \quad (47)$$

$$H(\alpha_1, \beta_1) > H(\alpha_1^{(0)}, \beta_1^*) \quad (48)$$

and a pair of inequalities

$$K(\alpha_1, \beta_1) > K(\alpha_1^{(0)}, \beta_1^*), \quad (49)$$

$$H(\alpha_1, \beta_1) \geqslant H(\alpha_1^{(0)}, \beta_1^*). \quad (50)$$

If pairs (47), (48) and (49), (50) are impossible then situation $\left\{\alpha_1^{(0)}, \beta_1^*\right\}$ must be efficient. Therefore, the supposition about Pareto-efficiency of situation (42) is contradictory.

Second, let stack

$$\left\{\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}, \left\{\alpha_2^*, \beta_2^*\right\}\right\} \tag{51}$$

be a Pareto-efficient situation by $\alpha_1^{(0)} \neq \alpha_1^*$ and $\beta_1^{(0)} \neq \beta_1^*$. This implies that both a pair of inequalities

$$K(\alpha_1, \beta_1) + K(\alpha_2, \beta_2) \geqslant K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + K\left(\alpha_2^*, \beta_2^*\right) \tag{52}$$

and

$$H(\alpha_1, \beta_1) + H(\alpha_2, \beta_2) > H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + H\left(\alpha_2^*, \beta_2^*\right) \tag{53}$$

and a pair of inequalities

$$K(\alpha_1, \beta_1) + K(\alpha_2, \beta_2) > K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + K\left(\alpha_2^*, \beta_2^*\right) \tag{54}$$

and

$$H(\alpha_1, \beta_1) + H(\alpha_2, \beta_2) \geqslant H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + H\left(\alpha_2^*, \beta_2^*\right). \tag{55}$$

Plugging $\alpha_2 = \alpha_2^*$ and $\beta_2 = \beta_2^*$ in the left sides of inequalities (52) — (55) gives a pair of inequalities

$$K(\alpha_1, \beta_1) \geqslant K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right), \tag{56}$$

$$H(\alpha_1, \beta_1) > H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) \tag{57}$$

and a pair of inequalities

$$K(\alpha_1, \beta_1) > K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right), \tag{58}$$

$$H(\alpha_1, \beta_1) \geqslant H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right). \tag{59}$$

If pairs (56), (57) and (58), (59) are impossible then situation $\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}$ must be efficient. Therefore, the supposition about Pareto-efficiency of situation (51) is also contradictory.

The Pareto-efficiency impossibility of other versions of 2-subinterval stacks is proved symmetrically. The Pareto-efficiency impossibility of N -subinterval stacks by $N \geqslant 3$ is proved similarly by ascending induction. □

So, if each of the subinterval 2-person games has a single Pareto-efficient solution, Theorem 2 allows finding the Pareto-efficient solution of the respective 2-person staircase-function game in a very simple way, just by stacking the subinterval solutions. It is easy to see that the assertion of Theorem 2 is reversible.

Theorem 3. If a 2-person staircase-function game (12) has a single Pareto-efficient situation, then each of the respective N games (19) by (14) — (18) and (20) — (25) has a single Pareto-efficient situation.

Proof. Let stack $\left\{\left\{\alpha_i^*, \beta_i^*\right\}\right\}_{i=1}^{N}$ be a single Pareto-efficient situation in a 2-person staircase-function game (12). This implies that both the pair of inequalities (38), (39) and the pair of inequalities (40), (41) are impossible for any $\alpha_i \in \left[a_{\min}; a_{\max}\right]$ and $\beta_i \in \left[b_{\min}; b_{\max}\right]$. Plugging $\alpha_k = \alpha_k^*$ and $\beta_k = \beta_k^*$ $\forall k = \overline{2, N}$ in the left sides of inequalities (38) — (41) gives a pair of inequalities

$$K\left(\alpha_1, \beta_1\right) \geqslant K\left(\alpha_1^*, \beta_1^*\right), \tag{60}$$

$$H\left(\alpha_1, \beta_1\right) > H\left(\alpha_1^*, \beta_1^*\right) \tag{61}$$

and a pair of inequalities

$$K\left(\alpha_1, \beta_1\right) > K\left(\alpha_1^*, \beta_1^*\right), \tag{62}$$

$$H\left(\alpha_1, \beta_1\right) \geqslant H\left(\alpha_1^*, \beta_1^*\right), \tag{63}$$

where the pair of (60) and (61) is impossible, and the pair of (62) and (63) is impossible as well. Hence, situation $\left\{\alpha_1^*, \beta_1^*\right\}$ is efficient. The efficiency of the remaining subinterval situations is proved in the same way.

Suppose that, along with efficient situation $\left\{\alpha_1^*, \beta_1^*\right\}$, situation $\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}$ is efficient also. Thus, neither the pair of inequalities (56), (57), nor the pair of inequalities (58), (59) is possible. Stack

$$\left\{\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}, \left\{\left\{\alpha_i^*, \beta_i^*\right\}\right\}_{i=2}^{N}\right\} \tag{64}$$

must not be efficient. This implies that a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + \sum_{i=2}^{N} K\left(\alpha_i, \beta_i\right) \leqslant K\left(\alpha_1^*, \beta_1^*\right) + \sum_{i=2}^{N} K\left(\alpha_i^*, \beta_i^*\right), \tag{65}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + \sum_{i=2}^{N} H\left(\alpha_i, \beta_i\right) < H\left(\alpha_1^*, \beta_1^*\right) + \sum_{i=2}^{N} H\left(\alpha_i^*, \beta_i^*\right) \tag{66}$$

holds or a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + \sum_{i=2}^{N} K\left(\alpha_i, \beta_i\right) < K\left(\alpha_1^*, \beta_1^*\right) + \sum_{i=2}^{N} K\left(\alpha_i^*, \beta_i^*\right), \tag{67}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + \sum_{i=2}^{N} H\left(\alpha_i, \beta_i\right) \leqslant H\left(\alpha_1^*, \beta_1^*\right) + \sum_{i=2}^{N} H\left(\alpha_i^*, \beta_i^*\right) \tag{68}$$

holds. Plugging $\alpha_k = \alpha_k^*$ and $\beta_k = \beta_k^*$ $\forall k = \overline{2, N}$ in the left sides of inequalities (65) — (68) gives a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) \leqslant K\left(\alpha_1^*, \beta_1^*\right), \tag{69}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) < H\left(\alpha_1^*, \beta_1^*\right) \tag{70}$$

and a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) < K\left(\alpha_1^*, \beta_1^*\right), \tag{71}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) \leqslant H\left(\alpha_1^*, \beta_1^*\right). \tag{72}$$

The possibility of either the pair of (69), (70) or the pair of (71), (72) means that situation $\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}$ is not efficient. Such a contradiction is similarly proved for any other subinterval situation.

Without losing generality, suppose that, along with efficient situations $\left\{\alpha_1^*, \beta_1^*\right\}$ and $\left\{\alpha_k^*, \beta_k^*\right\}$ by $k \in \left\{\overline{2, N}\right\}$, situations $\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}$ and $\left\{\alpha_k^{(0)}, \beta_k^{(0)}\right\}$ are efficient also. Then, anyway, stack (64) must not be efficient, which is the contradiction due to (65) — (72). Such a contradiction is similarly proved for any other combinations of subinterval situations. □

So, Theorem 3 asserts that when a Pareto-efficient stack is single, it does directly mean that every subinterval 2-person game must have a single Pareto-efficient situation. The question about multiple Pareto-efficient stacks is cleared right below.

7. What a Pareto-efficient stack consists of

It is easy to show that a finite 2-person staircase-function game (12) may have multiple Pareto-efficient situations. For example, a game with 2-subinterval 2-staircased function-strategies at the first player and 2-subinterval 3-staircased function-strategies at the second player, represented with respective matrices

$$\mathbf{K}_1 = \begin{bmatrix} 9 & 5 & 2 \\ 5 & 5 & 8 \end{bmatrix}, \ \mathbf{K}_2 = \begin{bmatrix} 1 & 7 & 2 \\ 5 & 7 & 1 \end{bmatrix} \tag{73}$$

and

$$\mathbf{H}_1 = \begin{bmatrix} 8 & 5 & 0 \\ 1 & 14 & 3 \end{bmatrix}, \ \mathbf{H}_2 = \begin{bmatrix} 6 & 3 & 1 \\ 9 & 0 & 1 \end{bmatrix}, \tag{74}$$

has 3 Pareto-efficient situations. They are the stack of efficient situations with payoffs $\{9, 8\}$ and $\{7, 3\}$, the stack of efficient situations with payoffs $\{9, 8\}$ and $\{5, 9\}$, and the stack of efficient situations with payoffs $\{5, 14\}$ and $\{5, 9\}$. By the way, the stack of efficient situations with payoffs $\{5, 14\}$ and $\{7, 3\}$ is not an efficient situation. Indeed, whereas the efficient (stacked) situations produce payoffs $\{16, 11\}$, $\{14, 17\}$, $\{10, 23\}$, the non-efficient stack produce payoffs $\{12, 17\}$. Obviously, a continuous 2-person staircase-function game may have multiple Pareto-efficient situations as well.

Theorem 4. Any Pareto-efficient situation in a 2-person staircase-function game (12) is a stack of successive Pareto-efficient situations in games (19) by (14) — (18) and (20) — (25).

Proof. Let stack $\left\{\left\{\alpha_i^*, \beta_i^*\right\}\right\}_{i=1}^{N}$ be a Pareto-efficient situation in the respective 2-person staircase-function game (12), where $\left\{\alpha_i^*, \beta_i^*\right\}$ is a Pareto-efficient situation in the game on "stair" subinterval i. Suppose that situation $\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}$ is not efficient in game (19) on the first "stair" subinterval, but stack (64) is an efficient situation in staircase-function game (12). Then a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+\sum_{i=2}^{N} K\left(\alpha_i^*, \beta_i^*\right) > K\left(\alpha_1^*, \beta_1^*\right)+\sum_{i=2}^{N} K\left(\alpha_i^*, \beta_i^*\right) \tag{75}$$

and

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+\sum_{i=2}^{N} H\left(\alpha_i^*, \beta_i^*\right) < H\left(\alpha_1^*, \beta_1^*\right)+\sum_{i=2}^{N} H\left(\alpha_i^*, \beta_i^*\right) \tag{76}$$

must hold, or a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+\sum_{i=2}^{N} K\left(\alpha_i^*, \beta_i^*\right) < K\left(\alpha_1^*, \beta_1^*\right)+\sum_{i=2}^{N} K\left(\alpha_i^*, \beta_i^*\right) \tag{77}$$

and

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+\sum_{i=2}^{N} H\left(\alpha_i^*, \beta_i^*\right) > H\left(\alpha_1^*, \beta_1^*\right)+\sum_{i=2}^{N} H\left(\alpha_i^*, \beta_i^*\right) \tag{78}$$

must hold, or just

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+\sum_{i=2}^{N} K\left(\alpha_i^*, \beta_i^*\right) = K\left(\alpha_1^*, \beta_1^*\right)+\sum_{i=2}^{N} K\left(\alpha_i^*, \beta_i^*\right), \tag{79}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+\sum_{i=2}^{N} H\left(\alpha_i^*, \beta_i^*\right) = H\left(\alpha_1^*, \beta_1^*\right)+\sum_{i=2}^{N} H\left(\alpha_i^*, \beta_i^*\right). \tag{80}$$

Inequalities (75) — (78) give a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) > K\left(\alpha_1^*, \beta_1^*\right) \tag{81}$$

and (70) and a pair of inequalities (71) and

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) > H\left(\alpha_1^*, \beta_1^*\right). \tag{82}$$

The pair of inequalities (81), (70) or the pair of inequalities (71), (82) means that situation $\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}$ is efficient. Besides, the pair of equalities (79) and (80) gives a pair of equalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) = K\left(\alpha_1^*, \beta_1^*\right), \tag{83}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) = H\left(\alpha_1^*, \beta_1^*\right). \tag{84}$$

These contradictions implying that stack (64) cannot be efficient are similarly proved for any other subinterval situation.

Suppose now that situations $\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}$ and $\left\{\alpha_2^{(0)}, \beta_2^{(0)}\right\}$ are not efficient in the first two subinterval games, but stack

$$\left\{\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}, \left\{\alpha_2^{(0)}, \beta_2^{(0)}\right\}, \left\{\left\{\alpha_i^*, \beta_i^*\right\}\right\}_{i=3}^{N}\right\} \tag{85}$$

is an efficient situation in staircase-function game (12). Then a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + K\left(\alpha_2^{(0)}, \beta_2^{(0)}\right) + \sum_{i=3}^{N} K\left(\alpha_i^*, \beta_i^*\right) >$$

$$> K\left(\alpha_1^*, \beta_1^*\right) + K\left(\alpha_2^*, \beta_2^*\right) + \sum_{i=3}^{N} K\left(\alpha_i^*, \beta_i^*\right), \tag{86}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + H\left(\alpha_2^{(0)}, \beta_2^{(0)}\right) + \sum_{i=3}^{N} H\left(\alpha_i^*, \beta_i^*\right) <$$

$$< H\left(\alpha_1^*, \beta_1^*\right) + H\left(\alpha_2^*, \beta_2^*\right) + \sum_{i=3}^{N} H\left(\alpha_i^*, \beta_i^*\right) \tag{87}$$

must hold or a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + K\left(\alpha_2^{(0)}, \beta_2^{(0)}\right) + \sum_{i=3}^{N} K\left(\alpha_i^*, \beta_i^*\right) <$$

$$< K\left(\alpha_1^*, \beta_1^*\right) + K\left(\alpha_2^*, \beta_2^*\right) + \sum_{i=3}^{N} K\left(\alpha_i^*, \beta_i^*\right), \tag{88}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + H\left(\alpha_2^{(0)}, \beta_2^{(0)}\right) + \sum_{i=3}^{N} H\left(\alpha_i^*, \beta_i^*\right) >$$

$$> H\left(\alpha_1^*, \beta_1^*\right) + H\left(\alpha_2^*, \beta_2^*\right) + \sum_{i=3}^{N} H\left(\alpha_i^*, \beta_i^*\right) \tag{89}$$

must hold, or just

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + K\left(\alpha_2^{(0)}, \beta_2^{(0)}\right) + \sum_{i=3}^{N} K\left(\alpha_i^*, \beta_i^*\right) =$$

$$= K\left(\alpha_1^*, \beta_1^*\right) + K\left(\alpha_2^*, \beta_2^*\right) + \sum_{i=3}^{N} K\left(\alpha_i^*, \beta_i^*\right), \tag{90}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + H\left(\alpha_2^{(0)}, \beta_2^{(0)}\right) + \sum_{i=3}^{N} H\left(\alpha_i^*, \beta_i^*\right) =$$

$$= H\left(\alpha_1^*, \beta_1^*\right) + H\left(\alpha_2^*, \beta_2^*\right) + \sum_{i=3}^{N} H\left(\alpha_i^*, \beta_i^*\right). \tag{91}$$

Inequalities (86) — (89) give a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + K\left(\alpha_2^{(0)}, \beta_2^{(0)}\right) > K\left(\alpha_1^*, \beta_1^*\right) + K\left(\alpha_2^*, \beta_2^*\right), \tag{92}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right) + H\left(\alpha_2^{(0)}, \beta_2^{(0)}\right) < H\left(\alpha_1^*, \beta_1^*\right) + H\left(\alpha_2^*, \beta_2^*\right) \tag{93}$$

and a pair of inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+K\left(\alpha_2^{(0)}, \beta_2^{(0)}\right)<K\left(\alpha_1^*, \beta_1^*\right)+K\left(\alpha_2^*, \beta_2^*\right), \tag{94}$$

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+H\left(\alpha_2^{(0)}, \beta_2^{(0)}\right)>H\left(\alpha_1^*, \beta_1^*\right)+H\left(\alpha_2^*, \beta_2^*\right). \tag{95}$$

As either of situations $\left\{\alpha_1^{(0)}, \beta_1^{(0)}\right\}$ and $\left\{\alpha_2^{(0)}, \beta_2^{(0)}\right\}$ is not efficient, then either a pair of inequalities (69), (70), or a pair of inequalities (71), (72) holds, and either a pair of inequalities

$$K\left(\alpha_2^{(0)}, \beta_2^{(0)}\right)\leqslant K\left(\alpha_2^*, \beta_2^*\right), \tag{96}$$

$$H\left(\alpha_2^{(0)}, \beta_2^{(0)}\right)<H\left(\alpha_2^*, \beta_2^*\right), \tag{97}$$

or a pair of inequalities

$$K\left(\alpha_2^{(0)}, \beta_2^{(0)}\right)<K\left(\alpha_2^*, \beta_2^*\right), \tag{98}$$

$$H\left(\alpha_2^{(0)}, \beta_2^{(0)}\right)\leqslant H\left(\alpha_2^*, \beta_2^*\right) \tag{99}$$

holds. After summing up inequalities (69) and (96), (71) and (98), (70) and (97), (72) and (99) sidewise, there is a pair of true inequalities

$$K\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+K\left(\alpha_2^{(0)}, \beta_2^{(0)}\right)\leqslant K\left(\alpha_1^*, \beta_1^*\right)+K\left(\alpha_2^*, \beta_2^*\right) \tag{100}$$

and

$$H\left(\alpha_1^{(0)}, \beta_1^{(0)}\right)+H\left(\alpha_2^{(0)}, \beta_2^{(0)}\right)\leqslant H\left(\alpha_1^*, \beta_1^*\right)+H\left(\alpha_2^*, \beta_2^*\right) \tag{101}$$

contradicting both pairs (92), (93) and (94), (95). These contradictions implying that stack (85) cannot be efficient are similarly proved for any other two or more (by ascending induction) situations. □

It is worth to note that Theorem 4 does not mean that any stack of successive efficient situations will be efficient. However, Theorem 4 does mean that if every subinterval (continuous) game has a finite number of Pareto-efficient situations, then all the Pareto-efficient situations in the respective 2-person staircase-function game (12) can be determined by just running over all possible stacks (whose number is finite) and selecting such stacks

$$\left\{\left\{\alpha_i^*, \beta_i^*\right\}\right\}_{i=1}^{N}$$

for which both the pair of inequalities (38), (39) and the pair of inequalities (40), (41) are impossible for any $\alpha_i \in \left[a_{\min}; a_{\max}\right]$ and $\beta_i \in \left[b_{\min}; b_{\max}\right]$.

8. Solving a finite 2-person staircase-function game

In a finite 2-person staircase-function game, players (forcedly or deliberately) act within a finite subset of possible values of their pure strategies. That is, these values are

$$a_{\min}=a^{(0)}<a^{(1)}<a^{(2)}<\ldots<a^{(M-1)}<a^{(M)}=a_{\max} \tag{102}$$

and

$$b_{\min} = b^{(0)} < b^{(1)} < b^{(2)} < \ldots < b^{(Q-1)} < b^{(Q)} = b_{\max} \tag{103}$$

for the first and second players, respectively, where $M \in \mathbb{N}$ and $Q \in \mathbb{N}$ (i. e., the player's function-strategy must have at least two different values). Then the succession of N continuous games (19) by (14) — (18) and (20) — (25) becomes a succession of N bimatrix games

$$\left\langle \left\{ \left\{ a^{(m-1)} \right\}_{m=1}^{M+1}, \left\{ b^{(q-1)} \right\}_{q=1}^{Q+1} \right\}, \left\{ \mathbf{K}_i, \mathbf{H}_i \right\} \right\rangle \tag{104}$$

with first player's payoff matrices

$$\mathbf{K}_i = \left[k_{imq} \right]_{(M+1)\times(Q+1)}$$

whose elements are

$$k_{imq} = K\left(a^{(m-1)}, b^{(q-1)}\right) = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m-1)}, b^{(q-1)}, t\right) d\mu(t) \;\text{ for }\; i = \overline{1, N-1} \tag{105}$$

and

$$k_{Nmq} = K\left(a^{(m-1)}, b^{(q-1)}\right) = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m-1)}, b^{(q-1)}, t\right) d\mu(t), \tag{106}$$

and with second player's payoff matrices

$$\mathbf{H}_i = \left[h_{imq} \right]_{(M+1)\times(Q+1)}$$

whose elements are

$$h_{imq} = H\left(a^{(m-1)}, b^{(q-1)}\right) = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a^{(m-1)}, b^{(q-1)}, t\right) d\mu(t) \;\text{ for }\; i = \overline{1, N-1} \tag{107}$$

and

$$h_{Nmq} = H\left(a^{(m-1)}, b^{(q-1)}\right) = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a^{(m-1)}, b^{(q-1)}, t\right) d\mu(t), \tag{108}$$

for $m = \overline{1, M+1}$ and $q = \overline{1, Q+1}$.

Let $\left\{ \alpha^*_{ij_i}, \beta^*_{ij_i} \right\}$ be an efficient situation in bimatrix game (104), where $j_i \in \left\{ \overline{1, J_i} \right\}$ and $J_i \in \mathbb{N}$. So, bimatrix game (104) has J_i efficient situations. It is unknown whether "participation" of situation $\left\{ \alpha^*_{ij_i}, \beta^*_{ij_i} \right\}$ in a stack makes the stack efficient or not. Let

$$\left\{ \left\{ \alpha^*_{ij_i}, \beta^*_{ij_i} \right\} \right\}_{i=1}^{N} \tag{109}$$

be a stack in a 2-person staircase-function game, which is the succession of N

bimatrix games (104),

$$\alpha^{*}_{ij_i} \in \left\{a^{(m-1)}\right\}_{m=1}^{M+1},\ \beta^{*}_{ij_i} \in \left\{b^{(q-1)}\right\}_{q=1}^{Q+1}.$$

Thus, stack (109) produces payoffs

$$\left\{u_l^{*}, v_l^{*}\right\} = \left\{\sum_{i=1}^{N} K\left(\alpha^{*}_{ij_i}, \beta^{*}_{ij_i}\right), \sum_{i=1}^{N} H\left(\alpha^{*}_{ij_i}, \beta^{*}_{ij_i}\right)\right\} \text{ by } l = \overline{1, \prod_{i=1}^{N} J_i}. \tag{110}$$

Let L be the number of efficient stacks, where

$$L \in \left\{\overline{1, \prod_{i=1}^{N} J_i}\right\}.$$

It is worth to remember that the case of when $L=1$ is only possible if $J_i = 1$ $\forall i = \overline{1, N}$ (see Theorem 3). Without losing generality, presume that namely the first L payoffs in (110) are produced by the efficient stacks (for instance, this can be done after sorting all the possible stacks just by separating the efficient from the non-efficient stacks). The best efficient stack can be found by a method suggested in (Romanuke, 2018a):

$$l_* \in \arg\max_{l=\overline{1, L}} \sqrt{\left(\frac{u_l^{*} - \min\limits_{z=\overline{1, L}} u_z^{*}}{\max\limits_{z=\overline{1, L}} u_z^{*} - \min\limits_{z=\overline{1, L}} u_z^{*}}\right)^2 + \left(\frac{v_l^{*} - \min\limits_{z=\overline{1, L}} v_z^{*}}{\max\limits_{z=\overline{1, L}} v_z^{*} - \min\limits_{z=\overline{1, L}} v_z^{*}}\right)^2}, \tag{111}$$

so the l_*-th stack is the best and the respective efficient payoffs $\left\{u_{l_*}^{*}, v_{l_*}^{*}\right\}$ are the most appropriate. Indeed, in terms of 0-1-standardization, they are the farthest from the zero payoffs $\{0, 0\}$ (the most unprofitable payoffs).

Consider an example case in which $t \in [0.8\pi; 2.8\pi]$, the set of pure strategies of the first player is

$$X = \left\{x(t),\, t \in [0.8\pi; 2.8\pi] : x(t) \in \{1+m\}_{m=1}^{7} \subset [2; 8]\right\} \subset \mathbb{L}_2[0.8\pi; 2.8\pi] \tag{112}$$

and the set of pure strategies of the second player is

$$Y = \left\{y(t),\, t \in [0.8\pi; 2.8\pi] : y(t) \in \left\{4.5 + 0.5\cdot(q-1)\right\}_{q=1}^{5} \subset [4.5; 6.5]\right\} \subset$$
$$\subset \mathbb{L}_2[0.8\pi; 2.8\pi], \tag{113}$$

where values of the pure strategies can change only at time points

$$\left\{\tau^{(i)}\right\}_{i=1}^{9} = \left\{0.8\pi + 0.2i\pi\right\}_{i=1}^{9}. \tag{114}$$

The first player's payoff functional is

$$K(x(t), y(t)) = \int\limits_{[0.8\pi; 2.8\pi]} \sin\left(0.65xyt - \frac{7\pi}{9}\right) e^{0.05xt}\, d\mu(t) \tag{115}$$

and the second player's payoff functional is

$$H\left(x(t),\, y(t)\right) = \int\limits_{[0.8\pi;\, 2.8\pi]} \sin\left(0.15xyt + \frac{6\pi}{7}\right) e^{0.01yt} d\mu(t). \tag{116}$$

Consequently, this game can be thought of as it is defined on rectangular lattice

$$\left\{1+m\right\}_{m=1}^{7} \times \left\{4.5+0.5\cdot(q-1)\right\}_{q=1}^{5} \subset [2;\, 8]\times[4.5;\, 6.5], \tag{117}$$

that is this game is a succession of 10 finite 7×5 (bimatrix) games

$$\left\langle \left\{ \left\{a^{(m-1)}\right\}_{m=1}^{7}, \left\{b^{(q-1)}\right\}_{q=1}^{5} \right\}, \left\{\mathbf{K}_i, \mathbf{H}_i\right\} \right\rangle =$$

$$= \left\langle \left\{ \left\{1+m\right\}_{m=1}^{7}, \left\{4.5+0.5\cdot(q-1)\right\}_{q=1}^{5} \right\}, \left\{\mathbf{K}_i, \mathbf{H}_i\right\} \right\rangle \tag{118}$$

with first player's payoff matrices

$$\left\{ \mathbf{K}_i = \left[k_{imq}\right]_{7\times5} \right\}_{i=1}^{10}$$

whose elements are

$$k_{imq} = \int\limits_{\left[0.8\pi+0.2\cdot(i-1)\pi;\, 0.8\pi+0.2i\pi\right)} f\left(a^{(m-1)}, b^{(q-1)}, t\right) d\mu(t) =$$

$$= \int\limits_{\left[0.8\pi+0.2\cdot(i-1)\pi;\, 0.8\pi+0.2i\pi\right)} f\left(1+m,\, 4.5+0.5\cdot(q-1),\, t\right) d\mu(t) =$$

$$= \int\limits_{\left[0.8\pi+0.2\cdot(i-1)\pi;\, 0.8\pi+0.2i\pi\right)} \sin\left(0.65\cdot(1+m)(4+0.5q)t - \frac{7\pi}{9}\right) e^{0.05(1+m)t} d\mu(t)$$

for $i = \overline{1, 9}$ (119)

and

$$k_{10mq} = \int\limits_{[2.6\pi;\, 2.8\pi]} \sin\left(0.65\cdot(1+m)(4+0.5q)t - \frac{7\pi}{9}\right) e^{0.05(1+m)t} d\mu(t), \tag{120}$$

and with second player's payoff matrices

$$\left\{ \mathbf{H}_i = \left[h_{imq}\right]_{7\times5} \right\}_{i=1}^{10}$$

whose elements are

$$h_{imq} = \int\limits_{\left[0.8\pi+0.2\cdot(i-1)\pi;\, 0.8\pi+0.2i\pi\right)} g\left(a^{(m-1)}, b^{(q-1)}, t\right) d\mu(t) =$$

$$= \int\limits_{\left[0.8\pi+0.2\cdot(i-1)\pi;\, 0.8\pi+0.2i\pi\right)} g\left(1+m,\, 4.5+0.5\cdot(q-1),\, t\right) d\mu(t) =$$

$$= \int\limits_{\left[0.8\pi+0.2\cdot(i-1)\pi;\, 0.8\pi+0.2i\pi\right)} \sin\left(0.15\cdot(1+m)(4+0.5q)t + \frac{6\pi}{7}\right) e^{0.01(4+0.5q)t} d\mu(t)$$

for $i = \overline{1, 9}$ (121)

and

$$h_{10mq} = \int\limits_{[2.6\pi;\,2.8\pi]} \sin\left(0.15 \cdot (1+m)(4+0.5q)t + \frac{6\pi}{7}\right) e^{0.01(4+0.5q)t} d\mu(t). \quad (122)$$

The 10 bimatrix 7×5 games (118) with (119) — (122) have 3, 1, 2, 3, 1, 4, 5, 4, 4, 5 Pareto-efficient situations, respectively. Therefore, there are 28800 stacks of such situations. The respective 2-person staircase-function game by (112) — (116) has 67 Pareto-efficient stacks presented in Figure 1, wherein a scatter plot (or, rather a cloud) of 28800 stack payoffs

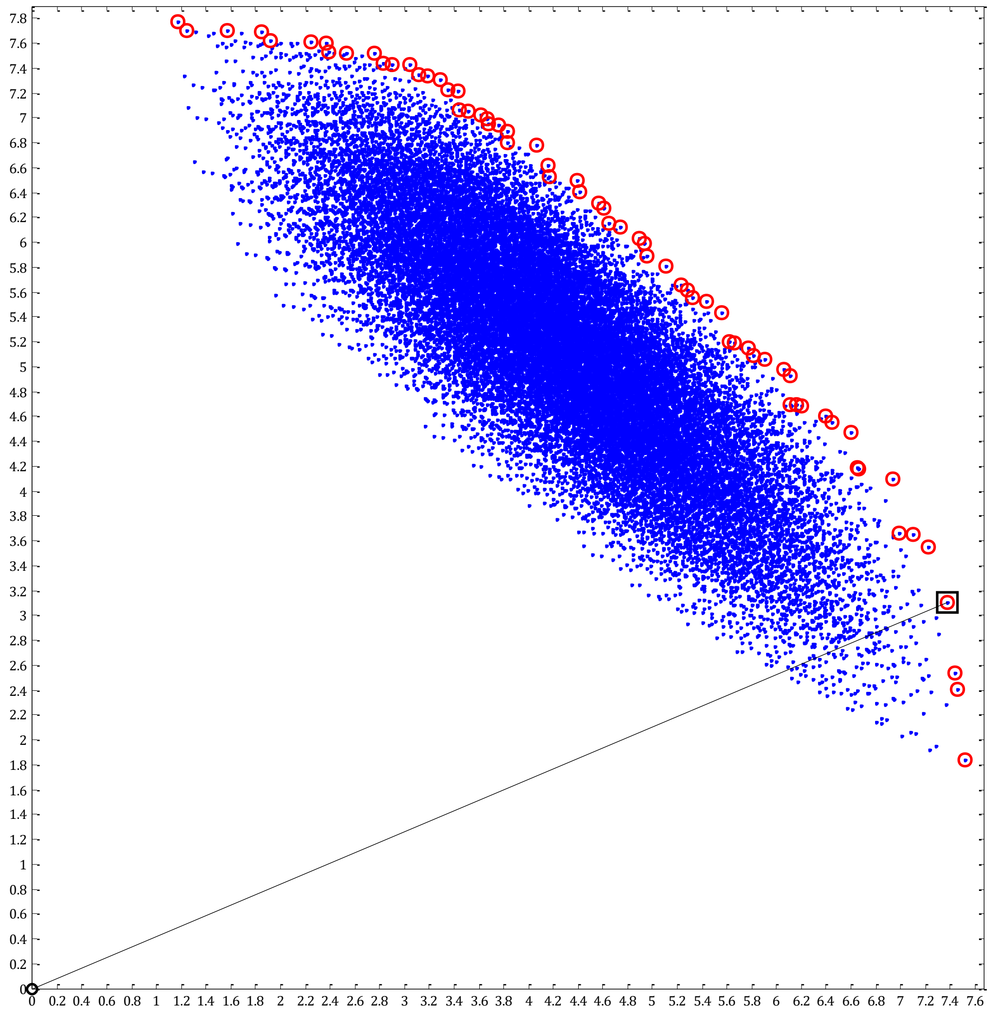

Figure 1. A scatter plot of 28800 stack payoffs in the finite 2-person staircase-function game and 67 payoffs (circles) by the efficient stacks (the best efficient payoffs point is squared)

$$\left\{u_l^*, v_l^*\right\} = \left\{\sum_{i=1}^{10} K\left(\alpha_{ij_i}^*, \beta_{ij_i}^*\right), \sum_{i=1}^{10} H\left(\alpha_{ij_i}^*, \beta_{ij_i}^*\right)\right\} \text{ by } l = \overline{1, 28800}$$

can be seen as well. The single best efficient payoffs point calculated by (111) as

$$l_* \in \arg\max_{l=\overline{1, 67}} \sqrt{\left(\frac{u_l^* - \min\limits_{z=\overline{1, 67}} u_z^*}{\max\limits_{z=\overline{1, 67}} u_z^* - \min\limits_{z=\overline{1, 67}} u_z^*}\right)^2 + \left(\frac{v_l^* - \min\limits_{z=\overline{1, 67}} v_z^*}{\max\limits_{z=\overline{1, 67}} v_z^* - \min\limits_{z=\overline{1, 67}} v_z^*}\right)^2}$$

corresponds to the best Pareto-efficient situation, whose first player's strategy $x^*(t)$ is shown in Figure 2 and second player's strategy $y^*(t)$ is shown in Figure 3. The best efficient payoffs are

$$\left\{u_{l_*}^*, v_{l_*}^*\right\} = \left\{u_1^*, v_1^*\right\} = \left\{7.376179, 3.100085\right\}.$$

Note that $l_* = 1$ is just an occasion of that among those 67 Pareto-efficient situations the best one appeared to be first. They are not sorted in any order.

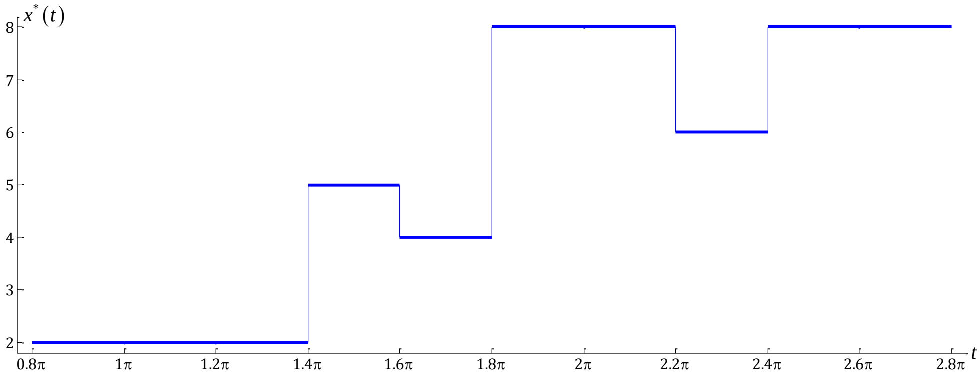

Figure 2. The best efficient staircase-function strategy of the first player

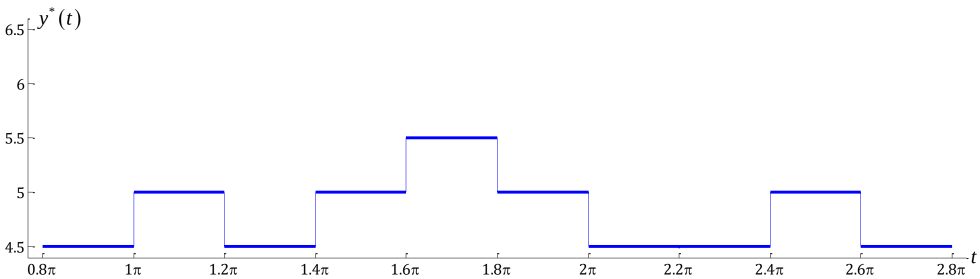

Figure 3. The best efficient staircase-function strategy of the second player

The example clearly shows that seeking for the efficiency in a finite noncooperative 2-person staircase-function game is an extremely hard task, which is only possible to accomplish by considering the respective succession of bimatrix games. Another, concomitant, task is the selection of the best Pareto-efficient situation among L

Pareto-efficient situations. This concomitant task exists when the conditions of Theorem 2 do not hold.

9. Discussion

Surely, in most cases, a subinterval 2-person game will have multiple Pareto-efficient situations. This does not diminish the value of Theorem 2. Its proof, by the way, directly follows from Theorem 4. The assertion of Theorem 3 is more peculiar and has a definite practical impact: if it is known that a staircase-function game has a single Pareto-efficient situation then its search is organized by the principle of the early stop — once an efficient situation in a subinterval 2-person game is found, the next subinterval game is solved.

So, the core of the method of solving 2-person games in staircase-function (finite or infinite, continuous) spaces consists in finding all Pareto-efficient situations in every subinterval 2-person game or proving that a subinterval game has a single efficient situation. The computation time depends on the "length" of the staircase-function game (i. e., on the number of subintervals). If the subinterval game is finite, its size influences the computation time also. The size is defined by the sets of possible values of players' pure strategies. In particular cases, obviously, solving a continuous subinterval game may cause considerable delay or be just intractable itself. Then the continuous subinterval game must be approximated with a finite (i. e., bimatrix) game using the known techniques (Romanuke & Kamburg, 2016).

Unfortunately, there is no universal method to finding all Pareto-efficient situations in a continuous (or infinite) 2-person game. Therefore, the finite approximation may become an intermediate in solving a staircase-function game. This is a factual transition to a finite staircase-function game.

The suggested method is a significant contribution to the mathematical 2-person game theory and practice for avoiding too complicated solutions resulting from game continuities, functional pure strategy spaces, and uncertainty in implementing equilibrium, profitability, fairness. The method is practically applicable owing to its tractability and simplicity. Owing to dealing with pure strategies only, it fits nonrepeatable games as well. Thus, the suggested method significantly simplifies 2-person games in staircase-function continuous and finite spaces by just "deeinstellungizing" them, similarly to preventing Einstellung effect in modeling (Loesche & Ionescu, 2020; Romanuke, 2020).

10. Conclusion

A finite noncooperative 2-person game in which strategies are staircase functions can be rendered to a bimatrix game, but it can hardly be solved straightforwardly due to a gigantic number of pure-strategy situations. The best way is to consider any 2-person staircase-function game as a succession of 2-person games in which strategies are constants. For a finite staircase-function game, each constant-strategy game is a bimatrix game whose size is sufficiently small to solve it in a reasonable time. For a continuous (infinite) staircase-function game, where the player has a continuum (infinity) of staircase function-strategies, each constant-strategy game is a classical continuous (infinite) 2-person game. Whichever the staircase-function game continuity is, any Pareto-efficient situation of staircase function-strategies is a stack of successive Pareto-efficient situations in the constant-strategy games (Theorem 4). The staircase-function game has a single Pareto-efficient situation if every constant-strategy game has a single Pareto-efficient situation (Theorem 2), and vice versa (Theorem 3). If a staircase-function game has two or more Pareto-efficient situations, the best efficient situation is found by holding it the farthest from the pair of the most unprofitable payoffs. This is fulfilled by just solving problem (111).

A similar question of finding the best efficient situation for games in staircase-function continuous and finite spaces should be studied for the case of three players. Then the presented assertions and conclusions might be just adapted to trimatrix and continuous (infinite) 3-person games, which model processes of practically optimizing the limited resources distribution among three sides as well as 2-person games do for two sides. Furthermore, the case with efficient Nash equilibria in staircase function-strategies would be quite interesting.

Funding: This research received no external funding.

Acknowledgments: This work was technically supported by the Faculty of Navigation and Naval Weapons, Polish Naval Academy, Poland.

Conflicts of Interest: The author declares no conflicts of interest.

References

Alva, S., & Manjunath, V. (2020). The impossibility of strategy-proof, Pareto efficient, and individually rational rules for fractional matching. Games and Economic Behavior, 119, 15–29.

Belhaiza, S., Audet, C., & Hansen, P. (2012). On proper refinement of Nash equilibria for bimatrix games. Automatica, 48 (2), 297–303.

Edwards, R. E. (1965). Functional Analysis: Theory and Applications. New York City, New York, USA: Holt, Rinehart and Winston.

Fu, H. (2021). On the existence of Pareto undominated mixed-strategy Nash equilibrium in normal-form games with infinite actions. Economics Letters, 201, 109771.

Harsanyi, J. C., & Selten, R. (1988). A general theory of equilibrium selection in games. Cambridge: The MIT Press.

Kayı, Ç., & Ramaekers, E. (2010). Characterizations of Pareto-efficient, fair, and strategy-proof allocation rules in queueing problems. Games and Economic Behavior, 68 (1), 220–232.

Ke, J., Li, X., Yang, H., & Yin, Y. (2021). Pareto-efficient solutions and regulations of congested ride-sourcing markets with heterogeneous demand and supply. Transportation Research Part E: Logistics and Transportation Review, 154, 102483.

Kim, S., Lee, Y. R., & Kim, M. K. (2019). Flexible risk control strategy based on multi-stage corrective action with energy storage system. International Journal of Electrical Power & Energy Systems, 110, 679–695.

Kontogiannis, S. C., Panagopoulou, P. N., & Spirakis, P. G. (2009). Polynomial algorithms for approximating Nash equilibria of bimatrix games. Theoretical Computer Science, 410 (19), 1599–1606.

Leyton-Brown, K., & Shoham, Y. (2008). Essentials of game theory: a concise, multidisciplinary introduction. San Rafael, CA: Morgan & Claypool Publishers.

Li, Y., Li, K., Xie, Y., Liu, J., Fu, C., & Liu, B. (2020). Optimized charging of lithium-ion battery for electric vehicles: Adaptive multistage constant current–constant voltage charging strategy. Renewable Energy, 146, 2688–2699.

Loesche, F., & Ionescu, T. (2020). Mindset and Einstellung Effect. Encyclopedia of Creativity. Cambridge, Massachusetts, USA: Academic Press, 174–178.

Moulin, H. (1981). Théorie des jeux pour l'économie et la politique. Paris: Hermann Paris.

Myerson, R. B. (1997). Game theory: analysis of conflict. Harvard: Harvard University Press.

Osborne, M. J. (2003). An introduction to game theory. Oxford: Oxford University Press.

Romanuke, V. V. (2018a). A couple of collective utility and minimum payoff parity loss rules for refining Nash equilibria in bimatrix games with asymmetric payoffs. Visnyk of Kremenchuk National University of Mykhaylo Ostrogradskyy, 1 (114), 38–43.

Romanuke, V. V. (2018b). Pure strategy Nash equilibria refinement in bimatrix games by using domination efficiency along with maximin and the superoptimality rule. Research Bulletin of the National Technical University of Ukraine "Kyiv Polytechnic Institute", 3, 42–52.

Romanuke, V. V., & Kamburg, V. G. (2016). Approximation of isomorphic infinite two-person noncooperative games via variously sampling the players' payoff functions and reshaping payoff matrices into bimatrix game. Applied Computer Systems, 20, 5–14.

Romanuke, V. V. (2020). Finite approximation of continuous noncooperative two-person games on a product of linear strategy functional spaces. Journal of Mathematics and Applications, 43, 123–138.

Schelling, T. C. (1980). The Strategy of Conflict. Harvard: Harvard University Press.

Vorob'yov, N. N. (1958). Situations of equilibrium in bimatrix games. Probability theory and its applications, 3, 318–331.

Vorob'yov, N. N. (1984). Game theory fundamentals. Noncooperative games. Moscow: Nauka.

Vorob'yov, N. N. (1985). Game theory for economist-cyberneticians. Moscow: Nauka.

Zheng, G., Fan, Q., Zhang, T., Zheng, W., Sun, J., Zhou, H., & Diao, Y. (2019). Multistage regulation strategy as a tool to control the vertical displacement of railway tracks placed over the building site of two overlapped shield tunnels. Tunnelling and Underground Space Technology, 83, 282–290.