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ON THE CONFORMITY OF SCALES OF MULTIDIMENSIONAL NORMALIZATION: AN APPLICATION FOR THE PROBLEMS OF DECISION MAKING

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Abstract: *The main goal of this paper is to harmonize the scales of normalized values of various attributes for multi-criteria decision-making models (MCDM). A class of models is considered in which the ranking of alternatives is performed based on the performance indicators of alternatives obtained by aggregating private attributes. The displacement of the domains of the normalized values of various attributes relative to each other and the local priorities of the alternatives are the main factors that change the rating when using various normalization methods. Three different linear transformations are proposed, which make it possible to bring the scales of normalized values of various attributes into conformity. The first transformation, the Reverse Sorting (ReS) algorithm, inverts the direction of optimization without displacing the areas of normalized values. The second transformation – IZ-method – allows researchers to align the boundaries of the domains of normalized values of various attributes in each range. The third transformation – MS-method – converts Z-scores into a sub-domain of the interval [0, 1] with the same mean values and the same variance values for all attributes. All transformations preserve the dispositions of the natural values of the attributes of the alternatives and ensure the equality of the contributions of various criteria to the performance indicator of the alternatives. The ReS-algorithm is universal for all normalization methods when converting cost attributes to benefit attributes. IZ and MS transformations expand the range of normalization methods in the case of using nonlinear functions aggregation of attributes.*

Key words: *Multiple criteria analysis; normalization; transformations of normalized values; conversion of measurement scales.*

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1. Introduction

Although partly formalized, Multi-Criteria Decision-Making (MCDM) methods are an important tool and have been successfully applied across multiple fields and disciplines (e.g., Hwang & Yoon, 1981; Tzeng & Huang, 2011). The specificity of MCDM tasks is that there is no access to reliable results for comparison. We obtain a rating only for candidate alternatives, and this rating depends to a large extent on the chosen aggregation method, the method of assigning criteria importance, and the normalization technique.

One of the unsolved problems of multi-criteria analysis is the choice of a method for normalizing the matrix of decisions. The main requirement of multivariate normalization as a method of data preprocessing is to ensure an equal contribution of each feature to the integral performance indicator of the alternative. The normalization process scales the criteria values to approximately the same value, however, different normalization methods may produce different solutions or ranking results. Most research concludes that the solution to the MCDM problem varies depending on the normalization method used. Numerous examples of comparative analysis presented in the literature when combining the normalization method and other parameters of the MCDM model (aggregation method, weights, distance metrics) confirm the impact of the choice of normalization method on the rating (Pavličić, 2001; Milani et al., 2005; Peldschus 2007; Migilinskas & Ustinovichius, 2007; Zavadskas et al., 2008; Ginevičius, 2008; Stanujkič, et al., 2013; Lakshmi & Venkatesan, 2014; Aouadni et al., 2017) et al. In some cases, this influence is significant.

An attempt to attribute such a result to the normalization method was not successful. It is not possible to single out the best or worst normalization method for a particular aggregation method. As a result, the direction of comparative analysis is still the main approach in MCDM problems when choosing a method for normalizing multidimensional data.

Research on the effectiveness of normalization methods has also been carried out in the field of object classification and machine learning, see for example, Singh D. & Singh B., (2020), Pandey & Jain (2017), Alshdaifat et al. (2021), Polatgil (2022), et al. In both MCMD and classification, normalization is a preliminary procedure integrated into the method. But unlike MCDM, in classification problems it is possible to select a set of best normalization methods based on classification efficiency criteria, while there is no efficiency criterion in MCDM. Nevertheless, as studies show, the superiority of normalization methods in classification problems is also relative and is largely determined by an applied problem with a well-defined data set.

In the absence of formalized criteria for choosing a normalization method, one approach recommends using the choice based on multiple voting (the Borda family of methods). Multiple voting determines the effective group of normalization methods that provide 1th rank (or 1-2, or 1-2-3) the most number of times for the same alternative. This allows, for example, to exclude from consideration methods of normalization that are not consistent with the majority.

Another line of research focuses on assessing the consistency of ratings obtained using different normalization methods for different feature aggregation methods Ranking consistency is assessed using various indicators: standard deviation as a measure of data scatter, Euclidean distance as a measure of closeness, Spearman's correlation; integral ranking consistency index (RCI) et al. (Chakraborty & Yeh, 2007, 2009; Çelen, 2014; Chatterjee & Chakraborty, 2014; Vafaei et al. 2018, 2022a, 2022b). A normalization method with a higher rating consistency across models is considered more reliable.

Given that MCDM models are partially formalized, and there is no concept of an absolutely optimal solution for them. Understanding this fact determines the recent trend in solving applied problems, when the solution is formed on the basis of a synthesis of estimates obtained using a certain reference set of models, including various procedures for aggregation, normalization, weighting, and other additional parameters (Palczewski & Salabum, 2019; Salabum et al., 2020; Rezk et al., 2021, 2022). The reference set of methods (including normalization methods) is formed on the basis of the selection of instances (methods), the use of which in solving applied problems had positive consequences.

An alternative to the comparative analysis and consistency of ratings in various MCDM models is the normalization method selection approach based on the principles that apply to data after normalization. The lack of criteria for choosing normalization methods is compensated by a certain set of basic principles that allow one to reject methods that lead to results that contradict these principles. In particular, Liping et al. (2009) recommend using the principle of "vertical" and "horizontal" normalization when choosing a normalization method to eliminate significant differences in the areas of normalized values.

Modern research uses more than 20 different normalization methods, both linear and non-linear. A fairly complete overview of the methods is presented in the works of Jahan & Edwards (2015) and Aytekin (2021). A feature of multivariate normalization is that the features are scaled on their individual scales and different normalization methods produce both a different range of values and their different density. In this case, the normalizations are not “isotropic,” that is, they compress the data cloud more in some directions and less in others. However, despite some violations of the data structure (mutual distances) this approach is generally accepted.

The normalized attribute values of the alternatives represent the share of the feature in different scales. In some cases, these shares may differ significantly, and the contribution of some attributes may dominate when they are aggregated into the performance indicator of alternatives. Therefore, a different range of normalized values of features and a shift in the areas of normalized values relative to each other leads to the priority of the contribution of individual criteria to the performance indicator of the alternatives. In particular, for linear normalization methods, this is formally due to the fact, that the compression ratios and the bias parameter depend on the measurement scale and the range of natural feature values.

The ranking result depends not only on the normalization method or one simple formula applied equally for all attributes but also on the relationship between the normalized values of various attributes. This problem is a feature of multidimensional data normalization and is not solvable.

Is it possible to influence such a situation? If the criteria for evaluating alternatives are independent, then the normalized values are also independent. This allows you to transform the range of normalized values and adjust the relationship between the domains and values of the normalized data.

In this study, the author promotes the approach of ensuring equal contribution of features after normalization using targeted linear transformation of the normalized values of different features. The preliminary material (Section 2) includes a description of some basic feature aggregation methods and linear decision matrix normalization methods.

Section 3 presents a number of invariant properties under the linear transformation of normalized values. Invariant properties guarantee the preservation of the information content of the original data during a linear transformation. Further given a meaningful interpretation of the main normalized scales, which is necessary for choosing a conditionally general scale when transforming the normalized values of all features.

Section 4 shows in detail the effect of shifting domains of normalized feature values relative to each other on the ranking results. The displacement of domains is formally related to the fact that the compression ratios and the displacement parameter depend on the scale of measurement and the range of natural values of features. Bias leads to the priority of the contribution of individual criteria to the performance indicator of alternatives. It is through purposeful transformation of normalized values that the elimination of bias and, accordingly, the priorities of the contribution of individual features to the performance indicator of alternatives is achieved.

The 5th section (the main result) presents three important transformations based on the fixed point technique that allow you to adjust the relationship between the domains and values of the normalized data.

The first transformation — Reverse Sorting (ReS) algorithm – transforms the area of normalized values of cost criteria into benefit criteria (performs the inversion of the direction of optimization) without shifting the areas of normalized values that occur when using data inversion of the form: $-r$, $1-r$, $1/r$.

The second transformation — IZ-method – allows to align the boundaries of the domains of normalized values of various attributes in a given range while preserving the dispositions of the attribute values of the alternatives.

When attributes are measured on different scales, they may be converted to Z-scores to aid comparison. The third transformation — MS-method – converts Z-scores to a sub-domain of the interval [0, 1] and stores the means, variances, and dispositions of the attribute values of the alternatives for each criterion.

Section 6 presents examples of solving MCDM problems with high sensitivity to variations in the decision matrix using the proposed transformations of normalized values for nonlinear aggregation methods WPM, WASPAS and COPRAS. This is followed by a conclusion.

2. Preliminaries

2.1. The MCDM rank model

In this study, a class of models in which the ranking of alternatives is performed based on the performance indicators of alternatives. The general description of such a model is as follows:

$$Q = F(A, C, D, \omega, nm, dm, pr). \tag{1}$$

The MCDM rank model includes the choice of a set of alternatives (A_i, i=1,...,m) and a set of criteria (C_j, j=1,...,n), an assessment of the values of the attributes of alternatives in the context of each criterion — a decision-making matrix D=(a_{ij}), definition, a method for assessing the weight or priority of criteria (w_j), a choice of a normalization method ($'nm'$) of decision making matrix, a choice of metric for calculating distances in n-dimensional space of criteria ($'dm'$), a choice of preference functions ($'pr'$), a definition of aggregation function (F) of the attributes of alternatives to calculate performance indicator (Q_i) of each alternative. Based on the calculation of the aggregate performance indicator of alternatives Q_i, alternatives are ranked.

Models of the form (2.1) are weakly formalized: the choice of A and C is determined by subjective preferences, the estimates of the decision matrix are not accurate, there are no criteria for choosing a method for evaluating the weights of criteria, a method for normalizing attributes, an aggregation method, a distance metric, a preference function, etc.

2.2. Aggregation models of attributes

This study uses two main approaches to aggregating the attributes of alternatives for which normalization matters.

In the first case, the aggregation is performed as a weighted sum of normalized attribute values in linear (SAW) or nonlinear versions (WPM, WASPAS, COPRAS):

- *SAW (Simple Additive Weighting) method or the WSM (Weighted Sum Model)* (e.g., Hwang & Yoon, 1981),

Performance indicator Q_i of the i-th alternative was determined as the entire standardized estimations of the attributes r_{ij} with the weight w_j of the j-th criteria:

$$Q_i = \sum_{j=1}^{n} w_j \cdot r_{ij} \ , \tag{2}$$

where $\sum_{j=1}^{n} w_j = 1$.

WPM (Weighted Product Model) (Hwang & Yoon, 1981),

Performance indicator Q_i of the i-th alternative was determined as:

$$Q_i = \prod_{j=1}^{n} (r_{ij})^{w_j} \ . \tag{3}$$

WPM method of aggregation is nonlinear.

- *WASPAS (Weighted Aggregated Sum Product Assessment)*,

WASPAS is a mixture (in proportion λ и (1-λ) between the weighted sum model (WSM) and the weighted product model (WPM) (Chakraborty & Zavadskas, 2014):

$$Q_i = \lambda \cdot \sum_{j=1}^{n} w_j \cdot r_{ij} + (1-\lambda) \cdot \prod_{j=1}^{n} (r_{ij})^{w_j} \ . \tag{4}$$

- *COPRAS (Complex Proportional Assessment) Method* (Zavadskas et al., 1994),

The aggregation method uses the construction of a performance indicator of alternatives based on the function of the two arguments S_{+i} and S_{-i}:

$$Q_i = S_{+i} + \frac{\sum_{i=1}^{m} S_{-i}}{S_{-i} \cdot \sum_{i=1}^{m} 1/S_{-i}} \ , \tag{5}$$

where

$$S_{+i} = \sum_{j=1}^{n} w_j \cdot r_{ij} \mid for\ j \in C_j^+,\ S_{-i} = \sum_{j=1}^{n} w_j \cdot r_{ij} \mid for\ j \in C_j^-. \tag{6}$$

Attribute aggregation using COPRAS is nonlinear across cost attributes.

Another group of aggregation methods uses the distance between data units (TOPSIS, GRA). When variables in a multidimensional dataset are at different scales, it makes sense to calculate distances to the ideal (or desired) object after some standardization:

- *TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution)* (e.g., Tzeng &Huang, 2011),

To determine the performance indicator of the *i*-th alternative Q_i, a homogeneous function was used:

$$Q_i = \frac{S_i^-}{S_i^+ + S_i^-}, \tag{7}$$

where

$$v_{ij} = r_{ij} \cdot w_j,\ S_i^+ = d(v_{ij}, v_j^+),\ S_i^- = d(v_{ij}, v_j^-), \tag{8}$$

$$v_j^+ = \{\max_i v_{ij} \,|if\ j \in C_j^+;\ \min_i v_{ij} \,|if\ j \in C_j^-\}, \tag{9}$$

$$v_j^- = \{\min_i v_{ij} \,|if\ j \in C_j^+;\ \max_i v_{ij} \,|if\ j \in C_j^-\}. \tag{10}$$

S_i^+ and S_i^- were the distances *d* between the ideal and anti-ideal objects respectively. Whereas the alternative A_i in the *n*-dimension attributes space, defined in one of the L_p-metrics. The TOPSIS ranking result depends on the choice of distance metric.

- *GRA (Grey Relation Analysis)* (e.g., Archana & Sujatha, 2012),

It evaluates the effectiveness of alternatives in two groups with respect to ideal and anti-ideal objects. The sequence of calculations is as follows:

Step 1: Define two sets of attributes i.e., ideal and anti-ideal:

$$r_j^{(1)} = \begin{cases} \max_i (r_{ij}),\ if\ j \in C_j^+ \\ \min_i (r_{ij}),\ if\ j \in C_j^- \end{cases},\quad r_j^{(2)} = \begin{cases} \min_i (r_{ij}),\ if\ j \in C_j^+ \\ \max_i (r_{ij}),\ if\ j \in C_j^- \end{cases}. \tag{11}$$

Step 2: Determine the matrix of deviations of normalized values from the ideal and anti-ideal:

$$V_{ij}^{(1)} = \left| r_j^{(1)} - r_{ij} \right|,\quad V_{ij}^{(2)} = \left| r_j^{(2)} - r_{ij} \right|. \tag{12}$$

Step 3: Determine the matrices the gray relational coefficient:

$$g_{ij}^{(1)} = \frac{\min_i \left(\min_j V_{ij}^{(1)} \right) + \beta \cdot \max_i \left(\max_j V_{ij}^{(1)} \right)}{V_{ij}^{(1)} + \beta \cdot \max_i \left(\max_j V_{ij}^{(1)} \right)}, \tag{13}$$

$$g_{ij}^{(2)} = \frac{\min_i\left(\min_j V_{ij}^{(2)}\right) + \beta \cdot \max_i\left(\max_j V_{ij}^{(2)}\right)}{V_{ij}^{(2)} + \beta \cdot \max_i\left(\max_j V_{ij}^{(2)}\right)}. \tag{14}$$

Step 4: Determination of the indicator performance for the alternative Q_i:

$$Q_i = Q_i^{(1)} / Q_i^{(2)}, \tag{15}$$

$$Q_i^{(1)} = \sum_{j=1}^{n} g_{ij}^{(1)} \cdot \omega_j,\ Q_i^{(2)} = \sum_{j=1}^{n} g_{ij}^{(2)} \cdot \omega_j. \tag{16}$$

In all of the above attribute aggregation models, the alternative with the most elevated Q_i score is considered the best.

2.3. Basic linear methods for normalization of decision matrix

For rank MCDM models, normalization is used, which produces values to the interval [0, 1]. 1 is the best value, 0 is the worst value, and intermediate values characterize the degree (proportion) of proximity to the best value. Conversion of features into normalized scales can be performed using various functions, both linear and nonlinear. This study focuses on linear methods of multidimensional normalization. The general form of linear methods of normalization of the decision matrix is as follows:

$$r_{ij} = \frac{a_{ij} - a_j^*}{k_j}, \tag{17}$$

where a_{ij}, r_{ij} are the natural and normalized values of the j-th attribute of the i-th alternative, respectively, a_j^* and k_j are some pre-assigned numbers, which we will call characteristic scales.

These numbers can be determined by statistical characteristics (normalization according to statistics), or given for some a priori considerations (normalization according to standards). Critical values of an indicator, best and worst “favorable” values, and other assessments lexically related to the problem of analysis can act as “standards”. Then these estimates have a subject interpretation. Table 1 presents 6 basic linear normalization methods most often used in multi-criteria choice problems (Chatterjee & Chakraborty, 2014; Vafaei et al., 2018; Jahan & Edwards, 2015; Aytekin, 2021).

Table 1. Basic linear methods for normalization of decision matrix.

non displacement: $r_{ij} = a_{ij} / k_j$			*with displacement:* $r_{ij} = (a_{ij} - a_j^*) / k_j$		
Max	**Sum**	**Vec**	**Max-Min**	**dSum**	**Z-score**
$k_j = a_j^{\max}$;	$k_j = \sum_{i=1}^{m} \lvert a_{ij} \rvert$;	$k_j = \sqrt{\sum_{i=1}^{m} a_{ij}^2}$	$k_j = a_j^{\max} - a_j^{\min}$; $a_j^* = a_j^{\min}$.	$k_j = \sum_{i=1}^{m} (a_j^{\max} - a_{ij})$; $a_j^* = a_j^{\max} - k_j$.	$k_j = std_i(a_{ij}) = s_j$ $a_j^* = mean_i(a_{ij})$

The short name of the normalization methods is determined by the semantic value of the compression ratio k. The method abbreviation is also used as the name of a function that converts values in accordance with the normalization method. For example, r_{ij}=Max(a_{ij})= a_{ij}/a_j^{max}. The **dSum** normalization method (Zeng et al., 2013) is an example of a multi-step procedure that is implemented by a combination of **Max-Min**, **Sum** and double inversion (Inv) normalized values:

$$r = \text{dSum}(a) = \text{Inv (Sum (Inv (Max-Min}(a)))) . \quad (18)$$

In details:

(1) t=Max-Min(a), Max–Min-method for all attributes,
(2) $u=1-t^+$, inversion values only for benefit attributes,
(3) v=Sum(u), Sum-method for all values u,
(4) $r=1-v$, inversion for all attributes.

For dSum-method, when normalizing cost attributes, the following reverse transform is used:

$$r_{ij} = 1 - \frac{a_{j*}^{\min} - a_{ij*}}{\sum_{i=1}^{m} (a_{j*}^{\min} - a_{ij})} . \quad (19)$$

3. Some important properties of linear normalization methods

3.1. Interpretation of normalized scales

A meaningful interpretation of normalized attribute values for basic linear normalization methods is as follows:

- **Max**: the fraction of the attribute of the i-th alternative relative to the largest attribute value or the degree of approximation to the best value 1,
- **Sum**: the proportion of the attribute of the i-th alternative from the total result (sum) or the contrast of the i-th alternative according to the j-th criterion $\sum r_{ij}=1$,
- **Vec**: the fraction of the attribute of the i-th alternative relative to the diameter of the m-dimensional rectangle built from the values of all alternatives or the projective angle, the equilibrium value of which is $1/\sqrt{m}$,
- **Max-Min**: the fraction equal to the ratio of the deviation of the attribute of the i-th alternative from the smallest value to the range of values of all alternatives according to the j-th criterion,
- **dSum**: inverted contrast of the maximum values of the attribute of the i-th alternative according to the j-th criterion,
- **Z-score**: the standardized deviation of the attribute of the i-th alternative from the mean of all alternatives, defined in multiples of the standard deviation.

Thus, the main methods of normalization have a well-defined agreement with the geometry of the value space or multidimensional cloud of the original data. However, the measurement scales and the geometry of the value space do not agree in any way. Obviously, compressing stretching and shifting the space of individual dimensions is not prohibited, since the attributes are independent. The stretch-compression ratios are different for different attributes, and the displacement ratios are also different. But in this case, it is necessary to justify the transformations and reconcile the normalized values of various attributes with each other in order to avoid unpredictability of results and consequences.

One of the reasons why the same normalization method is applied to different attributes is to interpret the normalized values of different attributes in the same way in order to subsequently aggregate values of the same order and not add the fractions of different values.

3.2. Invariant properties of linear normalization methods

P1. One of the main requirements for data normalization is the preservation of the information content of the data after transformation. It is reasonable to require that the proportions of the natural and normalized values of the attributes be preserved. The relative distance between the values of the *j*-th attribute of the *i*-th and *k*-th alternatives reduced to the range of the *j*-th attribute is defined as the disposition of the *i*-th and *k*-th alternatives by the *j*-th attribute:

$$d_{ik}^{j} = \frac{a_{ij} - a_{kj}}{a_j^{\max} - a_j^{\min}} . \tag{20}$$

It is easy to show that, for all linear methods of normalization, the dispositions between the natural and normalized values of the alternatives are preserved:

$$\text{if } r_{ij} = \frac{a_{ij} - a_j^*}{k_j} \Rightarrow \frac{r_{pj} - r_{qj}}{\mathrm{rng}(r_j)} = \frac{a_{pj} - a_{qj}}{\mathrm{rng}(a_j)},\ \forall\, p,q = \overline{1,m} . \tag{21}$$

Keeping the dispositions of alternatives after normalization means that the result of the uniform scaling is similar (in a geometric sense) to the original. But this only happens on the *j*-th attribute (coordinate). In the one-dimensional case, linear normalization methods are combinations of each other. There is a scaling of form under a linear transformation — namely, the invariance of the dispositions. "Scaling, a linear transformation that enlarges or diminishes objects" — https://en.wikipedia.org/wiki/Scaling. For multi-criteria problems, linear methods produce anisotropic scaling when at least one of the scaling factors is different from the others.

P2. Scaling with the same scale factor for each direction of the axis (scaling in the multidimensional case) does not affect the ranking result when the MCDM model uses a linear (SAW) or homogeneous function (TOPSIS, GRA) as the aggregation function.

The linear transformation $u_{ij} = k \cdot r_{ij} + b$ does not change the ranking if a linear function (SAW) is used to aggregate the attributes:

$$\text{if } Q_p(r) > Q_q(r) \text{ then for } u_{ij} = k \cdot r_{ij} + b \Rightarrow Q_p^*(u) > Q_q^*(u),\ \forall p,q = 1,\dots,m . \tag{22}$$

The performance indicators of alternatives are invariant with respect to the linear transformation in the case of using a homogeneous aggregation function (TOPSIS, GRA):

$$Q_p(r) = Q_p(u) . \tag{23}$$

P3. Normalization based on a linear transformation produces a measurement scale in which the first and second standardized moments (mean and standard deviation) are scaled by the same coefficients, and the third and fourth standardized moments (skewness and kurtosis) are invariants.

4. Ratios of normalized scales and variation in ranking

4.1. Domain of normalized values

The domain of normalized values for the above methods (except for the Z-score) is a subset of the set [0, 1]. Domain position to the j-th criterion on the set [0, 1] is determined by the parameters of the selected normalization method a_j^* and k_j. An illustration of the location of domains of normalized values for the main linear normalization methods is present in Figure 1.

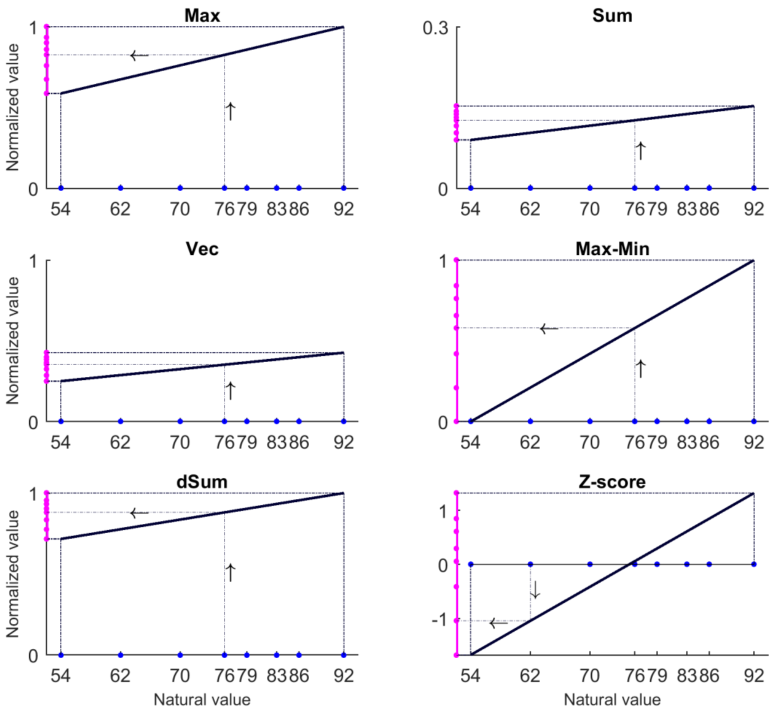

Figure 1. Domains for 2rd attribute when using different normalization methods. $a_{i,2}$=(54, 62, 86, 76, 83, 79, 92, 70).

For all linear normalization methods, the result of uniform scaling is similar (in the geometrical sense) to the original. With a linear transformation, the scaling of images takes place — the invariance of the arrangement of values.

However, in the multidimensional case, the scaling factors are different for different attributes. The stretch-contraction coefficients of the values for each j-th attribute for the Max-Min, Max, Vec and Sum normalization methods satisfy the inequalities:

$$a_j^{\max} - a_j^{\min} \overset{\text{for}\, a_j \geq 0}{\leq} a_j^{\max} \leq \left(\sum_{i=1}^{m} a_{ij}^2 \right)^{0.5} \leq \sum_{i=1}^{m} \left| a_{ij} \right| . \tag{24}$$

In this set (24), the compression coefficient for the dSum-method can take values from the second to the fifth position, and for the Z-score, the first and second positions, depending on the data distribution. Statistical experiment for evenly distributed over the interval [0, 1] data showed that in 90 and 67% of cases, respectively, inequality (24) for each attribute has the form:

$$k_{\mathrm{Z}} \overset{90\%}{\leq} k_{\text{Max-Min}} \overset{\text{for } a_j \geq 0}{\leq} k_{\mathrm{Max}} \leq k_{\mathrm{Vec}} \overset{67\%}{\leq} k_{dSum} \leq k_{\mathrm{Sum}}, \tag{25}$$

In particular (as shown in Figure 1), there will be an increase in the contribution of the “upper values” to the performance indicator of alternatives if the **dSum** or **Max** methods are used for the benefit attributes, or an increase in the contribution of the “smaller values” when using the **Sum** or **Vec** methods for the cost attributes.

4.2. Displacement of domains of normalized values of various attributes relative to each other

The essential difference between the linear methods in the multidimensional case is the mutual arrangement of the domains of the normalized values of various attributes relative to each other. Mutual arrangement of domains for various attributes on the set [0, 1] is determined by both absolute and relative values of the normalization parameters a_j^* and k_j attributes. A visual illustration of the relative position of the domains of normalized values for the main linear normalization methods is shown in Figure 2.

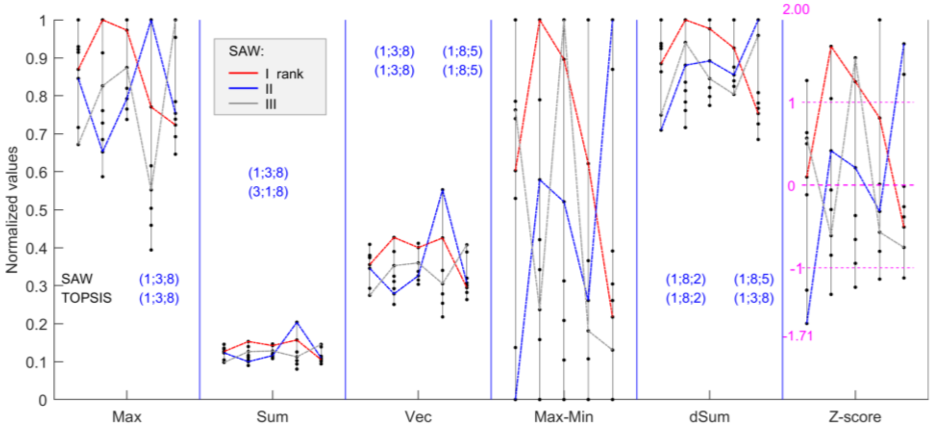

Figure 2. Mutual arrangement of domains of normalized values for the basic linear methods of normalization, and the result of ranking alternatives for the decision matrix D_0. Equal weight of criteria.

Hereinafter, the author uses the test case decision-making matrix D_0, dimension [8x5] (8 alternatives and 5 criteria), presented in the Table 2.

Table 2. Decision matrix (test case) $D_0 = (a_{ij})$ [8x5].

a_{ij}		Criteria				
		C_1^+	C_2^-	C_3^+	C_4^+	C_5^-
benefit(+)/cost(–)		+	–	+	+	–
Alternatives	A_1	580	54	178	2065	6000
	A_2	478	62	150	1056	4500
	A_3	564	86	145	2680	5800
	A_4	620	76	135	1230	5600
	A_5	615	83	183	1350	6200
	A_6	610	79	160	1650	5900
	A_7	667	92	140	1650	6500
	A_8	448	70	160	1480	4200

For each normalization method, the domain of each (of the five) attributes is represented by a vertical bar of values. The points represent an ordered set of normalized attribute values in the scale [0, 1]. The results for the Z-score normalization are given on an individual scale. Additionally, for each normalization method, the graph shows sequential numbers of alternatives of I-III ranks for SAW, TOPSIS aggregation methods (equal weights). Broken lines connect attribute values of ranks I, II, and III alternatives for the SAW aggregation method.

The jumps in the relative location of the normalized value domains of various attributes for different normalization methods are obvious. We observe, for example, a strong difference in the relative position of the 4th attribute domain. Only the Max-Min normalization method does not have its normalized value domains shifted.

1 rank has alternative A_1 (SAW) due to the greater contribution of the second, third and fourth attributes, and for the Sum normalization method, 1 rank is reversed. Alternative A_3 has 1 rank (TOPSIS) due to the priority on the fourth attribute (maximum value, blue line). There is also a reversion of alternatives of 2 and 3 ranks for normalization methods with a bias: Max-Min, dSum, Z-score. Figure 2 demonstrates that rank reversion is due to a shift in the domains of normalized values of various features relative to each other.

Thus, a different range of normalized values and a shift in the domains of normalized values of various attributes relative to each other leads to the priority of the contribution of individual criteria to the alternatives performance indicator, which entails changes in the ranking of alternatives.

Consequences:

• the **Sum** and **Vec** normalization methods should not be used for multivariate normalization or used only after additional bias analysis, as these methods have potentially large biases of different feature domains relative to each other. The **Sum** and **Vec** methods are good one-dimensional (vector) normalization methods that have an interpretation of the contribution intensity and projective angles,

• **Max** and **dSum** normalization methods equalize the upper values of all features (=1). For these methods, only the lower boundary of the domains is shifted. As a result, when choosing the best solution (when maximizing the integral indicator), the displacement of the lower levels has little effect on the result for the 1-rank alternative. However, as shown in the examples of Section 4.2, in the case of competition of alternatives, the rank inversion is also possible due to the displacement of the lower boundary of the regions,

• the **Max-Min** normalization method has no displacement in the areas of normalized values of various features (isotropic normalization), but the presence of zero values does not allow using it for some feature aggregation methods (WPM, WASPAS, COPRAS).

4.3. Local priorities of alternatives

In addition to displacement domains, ranking is influenced by the priorities of alternatives according to various criteria. Let us define the priorities of the alternatives according to various criteria as local priorities of the alternatives. If several, alternatives have local priorities according to different criteria, then a situation is possible in which the performance indicators of such alternatives will differ insignificantly. Therefore, to determine the priority of alternatives, it is not enough to compare the absolute values of the performance indicator Q_i according to (1). Attribute values may not be accurate. For example, an attribute can be measured approximately, the data source can be unreliable, the measurement was made in error, the measurements for various alternatives were carried out by different methods, some attributes can be random values or determined by interval values, etc. In such conditions, the solution is sensitive to error when evaluating the initial values of the attributes.

A situation with a high sensitivity of the solution can be recognized using the relative performance indicator of alternatives (relative PI):

$$\mathrm{d}Q_p = \frac{(Q_p - Q_{p+1})}{rng(Q)} \cdot 100\%, \quad p = 1, \ldots, m-1, \tag{26}$$

where Q_p is the value of the performance indicator corresponding to the p-rank alternative, $rng(Q)=Q_1-Q_m$.

dQ score is the relative (on a Q scale) increase or decrease in the performance score for an ordered list of alternatives. We believe that two alternatives, the relative increase in dQ of which differ less than the value of a given a priori error, should be considered indistinguishable.

In conditions of high sensitivity, the choice of the normalization method can change the value of the performance indicator, which will entail a change in the ranking of alternatives (Mukhametzyanov & Pamučar, 2018).

Let's demonstrate the above reasoning with an example. It is necessary to generate a decision matrix that is sensitive to the choice of the normalization method (with all other parameters being the same for the decision-making model).

The technique for generating such a matrix of solutions D_1 is based on the generation of random values (uniform distribution). It is necessary for each attributes to generate m random values (m alternatives) from the range of values determined by specifying the range of values. For each such decision matrix, ranking is performed using various normalization procedures (with other parameters of the MCDM model fixed). The iterative procedure for finding D_1 ends when, for the selected set of normalization methods, all the I-rank alternatives are different. In the Table 3 shows the decision matrix D_1 for which the 1-rank alternatives (TOPSIS aggregation) are different for the 6 basic linear normalization methods.

The decision matrix D_1 has the same range of values as the decision matrix D_0 in the Table 2.

Table 3. Decision matrix $D_1=(a_{ij})$, [8x5].

	C_1^+	C_2^-	C_3^+	C_4^+	C_5^-
A_1	667	70	135	1344	5669
A_2	478	66	154	2233	6234
A_3	448	71	156	2180	5188
A_4	506	54	183	1143	6500
A_5	455	87	163	1962	5901
A_6	531	92	137	2680	6282
A_7	558	65	181	1056	6457
A_8	531	67	136	1056	4200

The algorithm for generating the matrix D_1 is such that the range of the matrix D_1 and the relative position of the attribute domains correspond to the range of the matrix D_0. Such a decision matrix D_1 can hypothetically define the same decision-making problem as defined by specifying the matrix D_0, but with a different set of alternatives.

In Figure 3 shows the normalized values and the results of ranking alternatives for the decision matrix D_1.

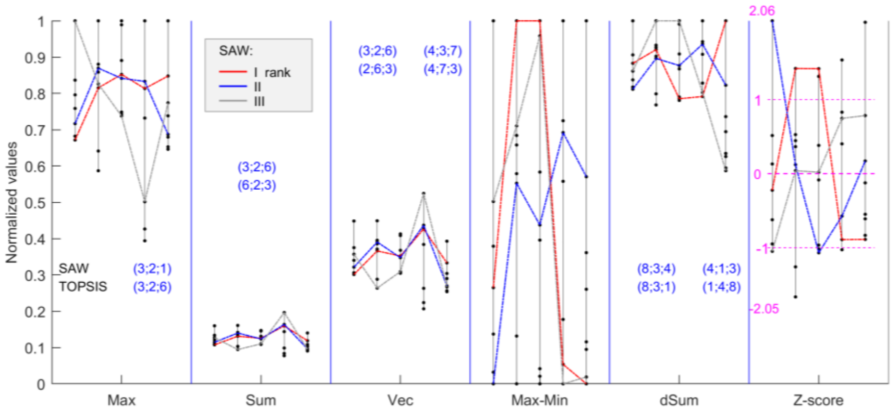

Figure 3. Mutual arrangement of areas of normalized values and ranking of alternatives for the decision matrix D_1 for various linear normalization methods (equal criteria weights).

In the title and comment of each option (subplot) in the Figure 3 includes the normalization method and the numbers of alternatives of I, II and III rank, respectively, obtained for six different aggregation methods — SAW and TOPSIS. Attribute aggregation is done with equal weights. The ranking results using various combinations of the normalization method and the aggregation method show that the alternatives of the 1-st rank for all 6 normalization methods when aggregated by the TOPSIS method are different for all 6 normalization methods. These are, respectively, alternatives A_3 (**Max**), A_6 (**Sum**), A_2 (**Vec**), A_4 (**Max-Min**), A_8 (**dSum**), A_1 (**Z-score**). When aggregating by the GRA-method, the alternative of the 1st rank for the 4 normalization methods (**Max**, **Vec**, **Sum** and **dSum**) are the same — A_3.

Figure 3 shows a situation in which some of the attributes are "strong" and approximately the same part is "weak". The performance indicators of such alternatives will differ slightly, and therefore the alternatives are hardly distinguishable. For the example presented (TOPSIS), the values for the relative performance indicator are presented in the Table 4.

Table 4. The relative performance indicator dQ. TOPSIS aggregation method.

Normalization method	A_i I rank	TOPSIS for D_0 dQ_I	TOPSIS for D_0 dQ_{II}	A_i I rank all different	TOPSIS for D_1 dQ_I	TOPSIS for D_1 dQ_{II}	A_i I rank all equal	GRA for D_1 dQ_I	GRA for D_1 dQ_{II}
Max	1	21,4	41,3	5	8,3	9,7	3	1,8	12,7
Sum	3	0,8	65,5	8	0,6	1,6	3	13,1	5,3
Vec	1	1,2	64,6	8	1,2	0,0	3	10,1	6,5
Max-Min	1	57,7	2,4	2	12,9	1,3	4	29,6	0,8
dSum	1	39,4	4,4	3	24,6	36,1	3	2,7	3,2
Z-score	1	62,8	0,2	7	6,4	2,2	4	22,7	6,9

Let us determine the value of a given a priori error (or accuracy) for calculating the performance indicator of alternatives Q, for example, 5%. Then the initial matrix of solutions D_0 when using the normalization methods **Max**, **Max-Min**, **dSum** and **Z-score** has a low sensitivity, and when using the methods of normalization **Sum** and **Vec** high sensitivity. The latter is due to the shift in the domains of the normalized values of all attributes in the interval [0, 1] for the **Sum** and **Vec** method, and flattening the upper values for the **Max**, **Max-Min**, **dSum** and **Z-score** normalization methods. The solution for D_0, when using all normalization options, demonstrates stability (Figure 3). In such a situation, the reliability of the result of ranking alternatives by the absolute value of the integral indicator Q is high.

For the decision matrix D_1, the relative difference in performance indicators for most normalization options does not exceed 1-5%, and in some cases the difference is fractions of a percent. In such a situation, the ranking of alternatives in terms of the performance indicator Q is doubtful. The solution is highly sensitive to the choice of the normalization method. Additionally, the solution is also highly sensitive to attribute estimates. A slight change in the values (in this example, enough in the second decimal place) leads to a change in the ranking.

For the variant where the ranking does not change the alternative with rank I (GRA), the relative difference in the performance indicators of almost all normalization options is more than 5%. The solution has little sensitivity to the choice of the normalization method.

Since the generated matrices have a similar range of values as for the matrix of solutions D_0 (see Table 2), the domains and their relative positions in the Figure 2 and Figure 3 are almost the same. This means that the ranking result depends on the ratio between the values of the original decision matrix and is determined by the local priorities of the alternatives for various attributes.

Similar results were obtained (generation of decision matrices D_1 and subsequent analysis of solutions) for the aggregation methods SAW, GRA, WPM, WASPAS and COPRAS.

When making decisions in situations of high sensitivity, several alternatives are generally recommended. But this becomes clear only after a comprehensive analysis.

Thus, it is necessary to consider that not only the normalization method determines the result, but also the relationship between the values of the original decision matrix or the local priorities of alternatives for various attributes.

5. Normalized data transformations

5.1. Transformation with a fixed-point technique

Casting attribute values (natural or normalized) by shifting to a new coordinate system with a zero initial value allows the origin to be used as a fixed point when scaling:

$$u_{ij} = a_{ij} - a_j^{\min}, \tag{27}$$

$$v_{ij} = r_{ij} - r_j^{\min}. \tag{28}$$

Such a procedure makes it possible to easily establish the necessary proportions between the scales of various attributes. In particular, the Max-Min normalization method has such an algorithm. In accordance with the calculation formula, the values are first shifted to a fixed point, and then scaling flattens the range of change of the normalized values of all attributes.

5.2. Inverting of optimization direction. ReS-algorithm

Two basic inverse transformations are used to convert the natural or normalized values of the cost criteria. These are 1) reflection relative to zero: $-r$, or $1-r$ with shift; 2) inverse transformation: $1/r$ (nonlinear inversion). All other inversions are functions of basic transformations.

The main problem when applying inverse transforms is shifting the range of values of attributes. This changes the contribution of the cost criteria to the aggregate measure of the effectiveness of the alternatives. Nonlinear inversion of the form $1/r$ preserves only the monotony of the values of the attributes of the alternatives. The ratios between the attribute values before and after the inversion change, which leads to a distortion of information in the source data. Transformation $1/r$ leads to a change in the range of normalized values and to a displacement in domains.

To normalize the cost features C_j^-, the ReS algorithm proposed by Mukhametzyanov (2020) is recommended:

$$\boxed{\begin{aligned} & r_{ij} = Norm(a_{ij}),\ \forall j = 1,\ldots,n, \\ & \tilde{r}_{ij^*} = -r_{ij^*} + r_{j^*}^{\max} + r_{j^*}^{\min},\ \forall j^* \in C_j^-. \end{aligned}} \tag{29}$$

The ReS-algorithm preserves the dispositions of natural and normalized values for cost attributes and the location of domains relative to each other:

$$\max_i(\tilde{r}_{ij^*}) = \max_i(r_{ij^*}),\ \ \min_i(\tilde{r}_{ij^*}) = \min_i(r_{ij^*}),\ \ \forall j^* \in C_j^-. \tag{30}$$

The ReS-algorithm allows the same linear normalization (Norm) method to be applied to the benefit and cost attributes and is versatile and efficient across all normalization methods.

An illustration of the inversion of the cost attributes of the matrix D_0 (2nd criterion) when using the Sum-method of normalization is shown in Figure 4.

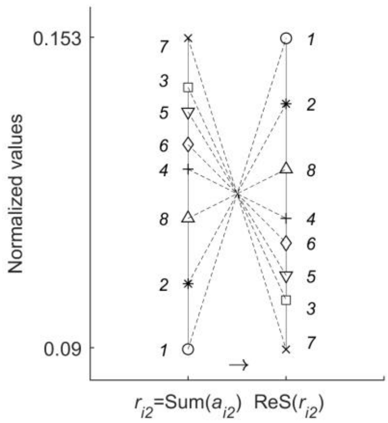

Figure 4. Graphic illustration of the inversion of normalized values (Sum) of cost criteria C_2^- to the benefit criteria.

The inversion of normalized values can also be done by changing the sequence of transformations. First, apply the ReS-algorithm for natural values of the attributes, then normalization with the chosen method:

$$a_{ij^*} = -a_{ij^*} + a_{j^*}^{\max} + a_{j^*}^{\min},\ \forall j^* \in C_j^-,\\ r_{ij} = Norm(a_{ij}),\ \forall j = 1,\dots,n. \quad (31)$$

In this case, the dispositions of natural and normalized values are preserved, however, there is a slight displacement in the domain and a change in the range of cost attributes relative to the normalized values obtained by the **Sum**, **Vec**, **dSum**, **Z-score** methods (except for **Max** and **Max-Min**), or one or both equalities (30) are fail.

The ReS-algorithm is a linear transformation and, as a consequence, preserves the dispositions of natural values (shape invariance).

The ReS-algorithm is a centrally symmetric transformation (see Fig. 5.1), so the mean and median are also transformed in a centrally symmetric way, the variance and excess do not change (invariant), the data skewness changes sign only.

5.2.1. Comparison of the ReS-algorithm of inversion and some normalization methods of cost criteria

A methodological error is the approach of separate normalization of cost criteria, which is used in almost all works on the normalization of multidimensional data. Data inversion should not be associated with the transformation of cost attributes into benefit attributes. The target value of an attribute in selection problems can be of three types: 1) smaller-the-better, 2) larger-the-better and 3) nominal-the best. Accordingly, criteria are refer to as cost criteria, benefit criteria and target nominal criteria. Coordination of the direction of the criteria is achieve by inverting the goal from a minimum to a maximum or vice versa. The choice of direction for maximizing or minimizing the performance indicator does not affect the ranking result. Inversion transforms the data in the following way: smaller values become larger, and vice versa, larger values become smaller. Considering the relativity of the direction and the independence of the solution from the change in the direction of optimization, the rational choice of which data to invert — for cost or benefit criteria, is determine by the ratio between the number of benefit and cost attributes in a particular problem. For example, if the task contains nine cost criteria and one benefit criterion (Rezk et al., 2021), then it is advisable to apply the inversion for the benefit criterion and search for the optimal solution by the criterion of minimizing the integral indicator.

Although it seems that the result of converting cost attributes to benefit attributes and data inversion is the same, this is not the case. Let's criticize a number of the most common formulas for converting cost attributes to benefit attributes for normalizing **Max**, **Sum** and **Vec** based on the transformation of the form 1–*r* (#1) and *Norm*(1/*a)* (#2) (Figure 5):

Poor inversion

		Max	Sum	Vec	
(#1)	Type 1	$r_{ij} = 1 - \frac{a_{ij}}{a_j^{\max}}$	$r_{ij} = 1 - \frac{a_{ij}}{\sum_{i=1}^{m} a_{ij}}$	$r_{ij} = 1 - \frac{1}{\sqrt{\sum_{i=1}^{m} a_{ij}^2}}$	✗
(#2)	Type 2	$r_{ij} = \frac{a_j^{\min}}{a_{ij}}$	$r_{ij} = \frac{1/a_{ij}}{\sum_{i=1}^{m} 1/a_{ij}}$	$r_{ij} = \frac{1/a_{ij}}{\sqrt{\sum_{i=1}^{m} 1/a_{ij}^2}}$	✗

Good inversion

(#3)	ReS	$r_{ij} = Norm(a_{ij}),\ \forall j = 1,\ldots,n$ $\tilde{r}_{ij^*} = -r_{ij^*} + r_{j^*}^{\max} + r_{j^*}^{\min},\ \forall j^* \in C_j^-$	✓

Figure 5. 1st and 2nd type of inversion vs. ReS-transform combined with Max, Sum, and Vec normalization methods.

Transformations of the 1st type (#1) lead to a shift (sometimes significant — antiphase as for *i*.Max1) of the domains of the normalized values of the cost criteria relative to the benefit, as soon as the average differs from 0.5. The natural value dispositions are preserved because the transformation is linear.

Transformations of the 2nd type (#2) are non-linear, therefore, they do not retain their shape (deform the mutual distances).

An illustration of the normalization-inversion in comparison with the ReS-transformation for the decision matrix according to the Table 2, where 2th and 5th criteria are cost criteria, is shown in Figure 6.

Figure 6 shows a strong bias (red lines) for the 1st type of transformation. This means that for the Max-method, the contribution of cost criteria (C_2 and C_5) will be greatly underestimated, and for the **Sum** and **Vec** normalization methods, on the contrary, the contribution will be overestimated.

It should not be surprising why a rank inversion would occur for this task. For example, in the SAW aggregation method (which requires the inversion of cost attribute values) for the problem under consideration, with a set of weight coefficients *w*=(0.17 0.19 0.17 0.17 0.3), a rank 1 inversion occurs from alternative A_4 at using type 2 inversion to alternative A_2 for 1st type of inversion.

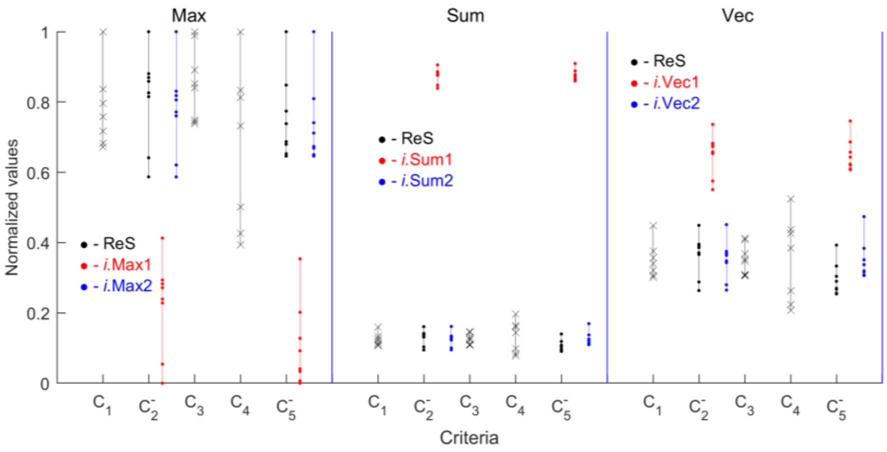

Figure 6. Domain displacement and shape deformation during inversion versus ReS-transformation. Decision matrix according to the Table 2. C_2^- and C_5^- care cost criteria.

The second type of inversion (#2) can be considered satisfactory. The domains of the normalized values are the same for the *i*.Max2 method or nearly the same for the *i*.Sum2 and *i*.Vec2 methods. However, the deformation of the form is not satisfactory (Figure 7, 8).

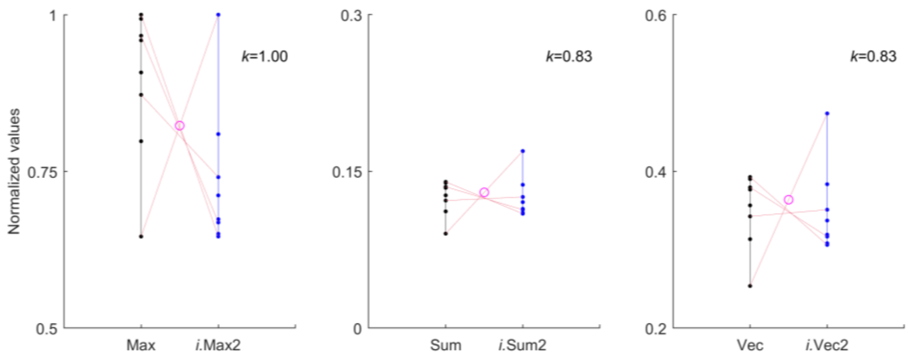

Figure 7. Data compression at inversion of the second type. Decision matrix according to the Table 2, 5th criterion (C_5^-).

The transformation does not have central symmetry. The degree of data compression (k) for the pair Sum-*i*.Sum2, Vec-*i*.Vec2 is approximately the same.

However, the mutual distances of normalized data and natural data are violated (shape deformation), which is well illustrated in Figure 8. Imagine a model of an architectural structure, for example, on a scale of 1:10. This model gives the observer a complete picture of the future structure. You can linearly transform the model and get another model, for example, on a scale of 1:100.

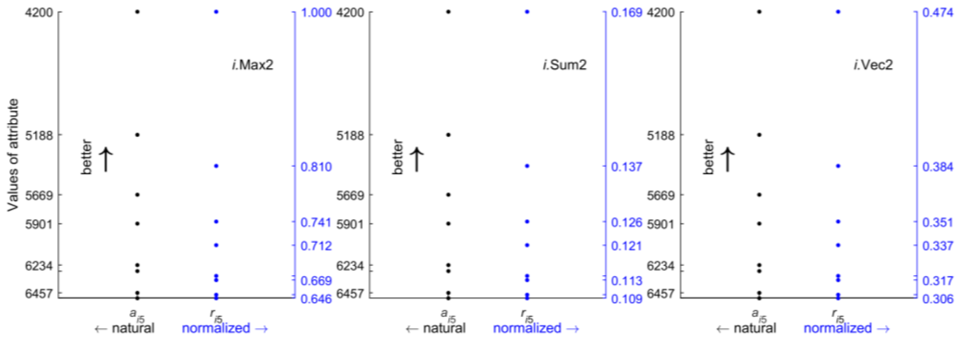

Figure 8. Violation of mutual distances during inversion of the second type. Decision matrix according to the Table 2, 5th criterion (C_5^-).

The idea of the building will not change. However, if this model is deformed non-linearly, then these will be two different models.

If the nonlinear deformation is not so significant, then these two models will still be similar. This example shows that the second type of conversion can preserve the ranking. However, why use such a transformation when the ReS-algorithm is available, which is a universal method for inverting attribute values of all types, is combined with any normalization method and allows you to agree on the direction of optimization for both cost criteria and profit criteria. The result of such a transformation excludes the priority of the feature's contribution to the integral performance indicator of the alternative.

Thus, based on the analysis performed, the authors do not recommend using 1st and 2nd type of inversion in combination with the **Max**, **Sum** and **Vec** normalization methods (Figure 5).

Note also that the ReS-algorithm used in combination with the Max-Min normalization method is the same as the standard cost criteria inversion:

$$i.\text{Max-Min:}\ \ r_{ij}=\frac{a_{ij}-a_j^{\min}}{a_j^{\max}-a_j^{\min}}=1-\frac{a_j^{\max}-a_{ij}}{a_j^{\max}-a_j^{\min}}=\mathrm{ReS}(\text{Max-Min}(a_{ij}))\,. \tag{32}$$

The ReS-algorithm is a universal method for inverting attribute values of all types and allows you to agree on the direction of optimization. The properties of the ReS-transformation are described in more detail in the study Mukhametzyanov (2020).

5.3. Eliminate the displacement in domains of normalized values. IZ-method of reduction to a conditionally general scale.

In accordance with section 4, for all linear methods (except Max-Min) there is a different magnitude of the range and a shift in the domains of normalized values for various attributes (Figure 2). The degree of difference in the domains of various attributes is different for different normalization methods, and also depends on the attribute measurement scales.

The problem of shifting the area of normalized values of various attributes is solved by the IZ transformation method (Mukhametzyanov, 2021). The main idea of IZ transformation is to align the domains of normalized values of different attributes. The key feature of the IZ-method is the choice of a common scale of normalized values that is consistent for all attributes and has the same interpretation. This allows aggregating normalized values of the same order, rather than fractions of different values.

It is assumed that a well-defined normalization method $Norm(a_{ij})$ is chosen for a specific problem. Such a choice may be due to a specific meaningful interpretation of the normalized values representing different proportions (values), as described in section 2.3 above.

The lower (similarly, upper) limit of normalized values for n attributes can be different:

$$\{r_1^{\min}, r_2^{\min}, \ldots, r_n^{\min}\} = \{\min_i r_{i1}, \min_i r_{i2}, \ldots, \min_i r_{in}\}\ , \tag{33}$$

$$\{r_1^{\max}, r_2^{\max}, \ldots, r_n^{\max}\} = \{\max_i r_{i1}, \max_i r_{i2}, \ldots, \max_i r_{in}\}\ . \tag{34}$$

Let's perform the transformation of the normalized values of each attribute from the interval $[r_j^{\min}, r_j^{\max}]$ into some fixed interval $[I, Z] \subset [0, 1]$ using the fixed point technique. To do this, we will shift the domain to a new coordinate system with a zero initial value

$$r_{ij} - r_j^{\min}\ . \tag{35}$$

Next, we perform a linear transformation of the normalized values of all attributes into the interval $[I, Z]$ using stretch-compression and shear operations:

$$\boxed{z_{ij} = \frac{r_{ij} - r_j^{\min}}{r_j^{\max} - r_j^{\min}} \cdot (Z - I) + I,\ \forall i = \overline{1,m};\ \forall j = \overline{1,n}\ .} \tag{36}$$

As a result, we obtain new normalized values $z_{ij} \in [I, Z] \subset [0, 1]$.

A step-by-step illustration of the IZ-transformation of normalized values is shown in Figure 9.

Interval $[I, Z]$ represents a common, consistent scale of normalized values for all attributes. Limit values are defined, respectively, as:

$$I_{\min} = \min r_j^{\min},\ I_{\max} = \max r_j^{\min}\ , \tag{37}$$

$$Z_{\min} = \min r_j^{\max},\ Z_{\max} = \max r_j^{\max}\ . \tag{38}$$

In order to reduce the potential impact of a single attribute, averaging is used to match the scales. The following rational options are proposed for use:

(i) As I, take the average value of the lower level of attributes, and as the Z value, take the average value of the upper level

$$I_1 = r^{\min} = \operatorname{mean} r_j^{\min},\ Z_1 = \operatorname{mean} r_j^{\max}, I_1 < Z_1\ . \tag{39}$$

In this case, the scales agree within the standard deviation for the mean.

(ii) As I, take the median value of the lower level of attributes, and as the value of Z, take the median value of the upper level

$$I_2 = r^{\min} = \operatorname{median} r_j^{\min},\ Z_2 = \operatorname{median} r_j^{\max}, I_2 < Z_2\ . \tag{40}$$

In this case, the scales are consistent within the standard deviation of the median.

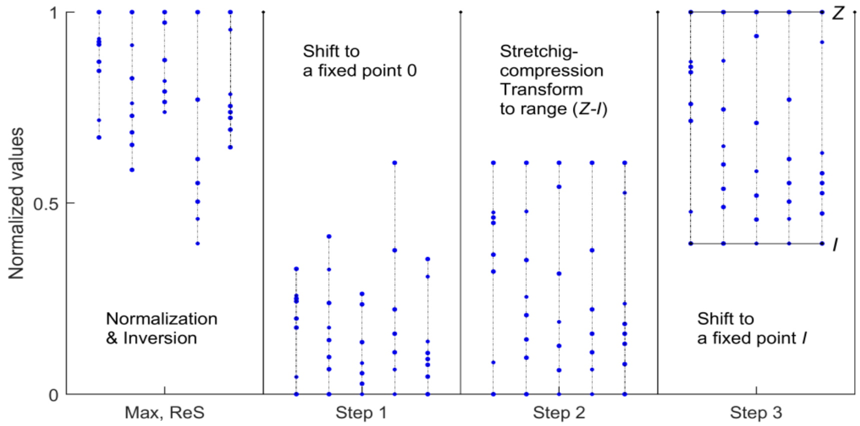

Figure 9. Step-by-step IZ-transform of normalized values (Max).

It is also possible to choose a choice, determined by the context of the decision-making problem, in which the interval [I, Z] is determined by the expert: $0 \le I_3 \le Z_3 \le 1$.

If as [I, Z] accept [0, 1], then IZ-normalizations are similar to the result of the Max-Min(r_{ij}) transformation.

An illustration of the transformation of normalized values using the IZ-method for various normalization methods for the choice case [I, Z] in accordance formula (40) is present in Figure 10. The graph additionally shows the results of aggregation of normalized attribute values (alternatives number's of I-III rank) using the SAW, TOPSIS(L_2), GRA, WPM, WASPAS and COPRAS aggregation methods. Attribute aggregation is done with equal weights.

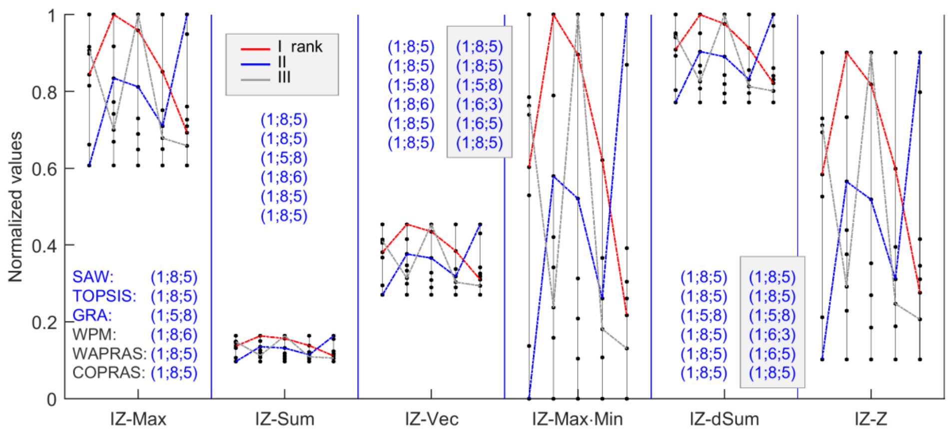

Figure 10. Transformation of normalized values using the IZ-method for various normalization methods. Decision matrix D_0. Aggregation of attributes with equal weights.

For the first three attribute aggregation methods (SAW, TOPSIS(L_2), GRA), the alternatives ranking results are the same for all normalization methods. According to invariant property 2, in the case of aggregating attributes using a linear or homogeneous function, the ranking results coincide with the ranking results when using the **Max-Min** normalization (Figure 10).

The WPM, WASPAS, and COPRAS methods use a non-linear aggregation function. In this case, the ranking results for different normalization methods may differ due to the use of different scales for measuring the normalized values (see, for example, the ranking results for normalization IZ-Max, IZ-dSum and IZ-Z presented in Figure 10. Applying an IZ transformation to any set of normalized values changes the data structure. The disposition and range of values corresponding to the normalization method are saved. Thus, the IZ-transformation (and the choice of the interval $[I, Z]$) is relevant only in the case of using non-linear aggregation methods, for example, WPM, WASPAS, COPRAS for rank-based MCDM models.

5.4. MS transformation of normalized values

When scores are measured on different scales, they may be converted to Z-scores to aid comparison. The Z-score produces normalized values in multiples of the standard deviation with a mean of 0. Therefore, the Z-score is used in many important applications to compare the attributes of objects in the same population. The appeal of Z standardization for solving MCDM problems is that in this case the domains of normalized values are equalized on average. This eliminates the priority of the contribution of individual attributes to the performance indicator of alternatives.

The observed values above the mean have positive standard points, while the values below the mean have negative standard points (see Figure 1), which in some cases contradicts the logic of data analysis in the multivariate case and is a drawback of standardized Z-scales. For example, when using WPM (3) attribute aggregation, negative standard points are not allowed.

Considering that the interval of normalized values during standardization includes both positive and negative values, when aggregating attributes, their compensation is possible (for example, for additive aggregation methods). However, if you transform the standardized values into the [0, 1] using linear transformations, then according to the invariant properties (see Section 3.2), both the disposition of attributes and the ranking of alternatives are preserved. For linear or homogeneous aggregation functions, the values of the performance indicators of the alternatives change strictly monotonically and the compensation of positive and negative values does not affect the ranking.

Below is an algorithm for linear transformation of standardized values using the fixed point technique — MS-method (Mean & Standard deviation):

Step 1. Perform the inversion (if necessary) of natural values of cost attributes using the ReS-algorithm.

Step 2. Perform the standardization

$$z_{ij} = \frac{a_{ij} - \overline{a}_j}{s_j} \quad , \tag{41}$$

where $\overline{a}_j$, s_j are the mean and standard deviation of the values of the j-th attribute, respectively.

Step 3. Shift the values to a fixed point

$$u_{ij} = z_{ij} - \min_j(\min_i(z_{ij})) \ . \tag{42}$$

Step 4. Stretch-compressing

$$v_{ij} = u_{ij} / \max_j(\max_i(u_{ij})) \ . \tag{43}$$

Step 5. We carry out the reduction of values to the scale of the normalization method

$$v_{ij} = v_{ij} \cdot k \ . \tag{44}$$

Step 6. Shift all values to 1 (top level)

$$v_{ij} = v_{ij} + 1 - \max_j(\max_i(v_{ij})) \ . \tag{45}$$

A step-by-step illustration of MS-transformation of normalized values is shown in Figure 11.

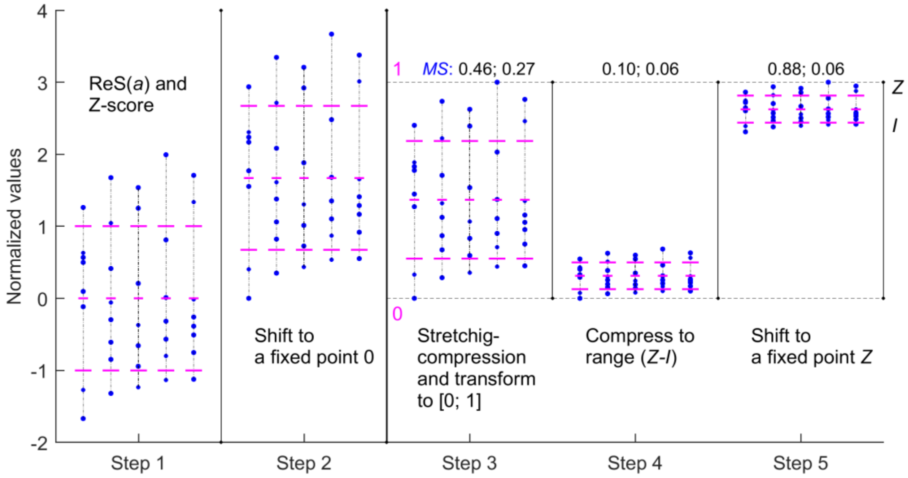

Figure 11. Step-by-step MS-transform.

The mean and standard deviation for all attributes are the same (in the Figure 11 shows footnotes of the values and the graphs show cut-offs in highlighted color). Step 4 transforms the values in [0, 1].

In step 5, the values are scaled to the scale of the normalization method (as in the case of the IZ-method). Therefore, the scale factor k ($0 < k \leq 1$) is defined similarly:

$$k = Z - I \ . \tag{46}$$

For k=1, the Z-score scale is valid.

When k=1, the scale of the Max-Min normalization method takes place. For other normalization scales, it is proposed to use similar options as for the IZ-transformation, defined by equations (37)–(40). An illustration of the transformation of normalized values using the MS-method for various options for choosing normalized scales for the case of choosing a scaling factor k by Eq. (5.20) is present in Figure 12. The graph additionally shows the results of aggregation of normalized attribute values (alternatives number's of I-III rank) using the SAW,

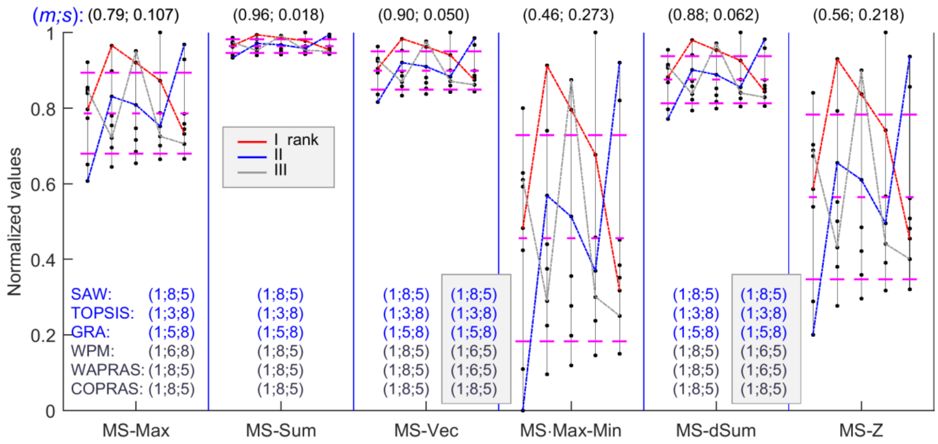

Figure 12. MS transformation using various normalized scales. Decision matrix D_0.

TOPSIS(L_2), GRA, WPM, WASPAS and COPRAS aggregation methods. Attribute aggregation is done with equal weights. The mean and standard deviation for all attributes is the same (in the Figure 12 shows footnotes of the values and in the plot, they are marked as cut-offs in highlighted color). For the first three methods, the ranking results according to the invariant property 2 do not depend on normalization. The COPRAS method uses a non-linear aggregation function for cost criteria. Considering that at the first step of the MS transformation, the values of the cost criteria are inverted, the results of aggregation by the COPRAS method will be like the SAW method.

Thus, when MS transformation mean values and variances are the same for all attributes; the areas of normalized values flatten on average. Additionally, the choice of the measurement scale is set in accordance with the selected method of normalization. Eliminating negative Z-scores allows you to expand the list of decision-making methods. For example, the use of WPM, WASPAS methods becomes acceptable.

The MS transformation (and the choice of the scaling factor k) is relevant only in the case of using nonlinear aggregation methods. When aggregating attributes using a linear or homogeneous function, the ranking results are the same as the ranking results when using Z-score normalization.

5.5. Anisotropic scaling using IZ and MS transforms

Applying IZ and MS transforms to any set of normalized values changes the range and relative position of attribute domains. Attribute dispositions within a domain are preserved. A multidimensional data cloud characterizing alternatives (objects) undergoes deformation in separate directions (measurement scales). A special case of such deformations for three attributes is shown in Figure 13 as a 3D scatter plot.

The illustration was made for a task with three attributes. The decision matrix was obtained from the D_0 matrix (Table 2) by excluding 2 and 4 attributes.

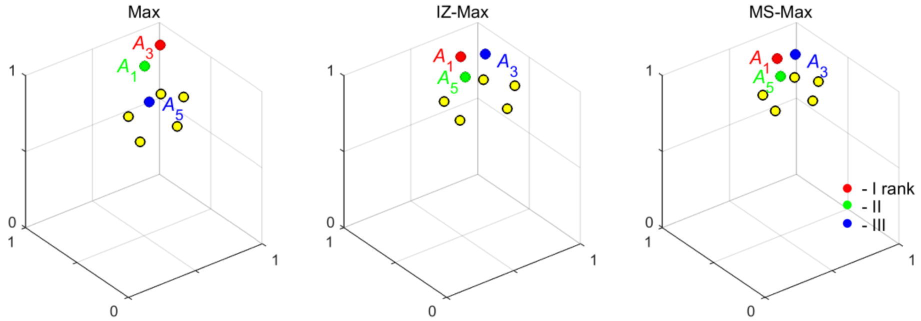

Figure 13. 3D illustration of IZ and MS transformations for 3 criteria. Decision matrix D_0, attributes 1, 3 and 5.

The use of IZ and MS transformations is determined by the task of harmonizing the measurement scales of individual attributes. The formal criterion for such a transformation is the elimination of the priority or contribution of individual attributes to the integral indicator of the alternative at the normalization stage. In the example shown (Figure 13), the IZ and MS transformations for Max normalization smooth out the slight warping that is seen in the multidimensional data cloud. This leads to a change in the ranking of alternatives (SAW aggregation method). After the transformation, the first rank changed from A_3 (Max) to A_1 (IZ-Max, MS-Max). For the case of more than 3 attributes, visualization of the data cloud is possible using a radar chart (Chambers et al., 2018). Radar charts are primarily suited for strikingly showing outliers, or when one chart is greater in every variable than another. Radar charts are mainly used for ordinal measurements, where each variable is "better" in some respect and all variables are measured on the same scale, corresponding to normalized data.

Visualization of the cloud of normalized and transformed data in the form of a radar diagram for the decision matrix D_0 is present in Figure 14.

The horizontal ray (angle 0^o) corresponds to the first attribute of the D_0 matrix. The countdown is counter clockwise. The area limited by the outer web is a characteristic of objects (alternatives) and correlates with the corresponding values of the integral indicator (for example, for the SAW method). The values of the reduced area in the Figure 15, are shown at the top of each fragment, and the rank of the alternative (SAW) is indicated in the lower left corner of the radar diagram.

Radar charts are not well suited for making trade-off decisions — when one chart is better than another in some variables, but worse in others, since the contained area becomes proportional to the square of linear measures and this area also depends on the order of the attributes. However, visualization based on the radar diagram makes it possible to assess the degree of configuration of the data cloud and assess the degree of its deformation after IZ and MS transformation. This allows you to purposefully eliminate the priority of the contribution of individual attributes to the integral indicator of the alternative at the normalization stage.

In the example shown (Figure 15), the IZ and MS transforms for Max normalization smooth out the slight warping that is seen in the multidimensional data cloud for alternatives A_5 and A_8.

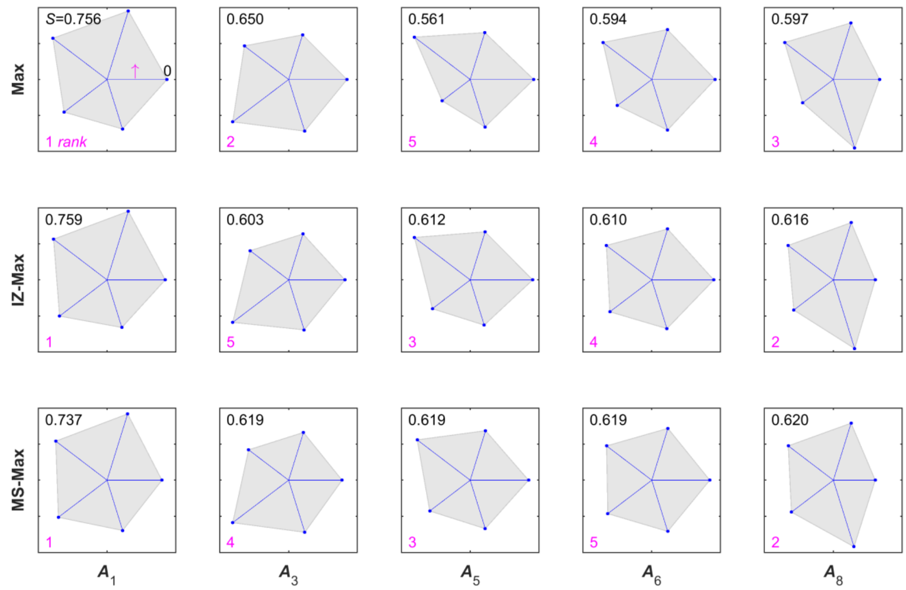

Figure 14. Deformation of a cloud of data of multidimensional objects based on a radar diagram for IZ and MS transformations. Decision matrix D_0. Equal weights. Fragments of the first five (by rating) alternatives.

This leads to a change in the ranking of alternatives (SAW-method of aggregation). After the transformation, the A_8 alternative becomes the second rank alternative, and the A_5 alternative becomes the third rank.

The use of weighting factors when aggregating attributes will obviously lead to deformation of the data cloud. Visualization of the influence of weights on the representation of the data cloud in the form of a radar diagram is carried out by adjusting the angles (sectors). In the case of equal weights (as in Figure 14), the angle of each sector is equal to $\varphi=2\cdot\pi/n$. For the case of different weights, the angles are calculated in proportion to the weight coefficients ($\Sigma\omega_j=1$).

$$\varphi_j=2\cdot\pi\cdot\omega_j/n. \tag{47}$$

The radar diagram for the decision matrix D_0 for the case of MS-Max transformation, taking into account the weighting coefficients, is present in Figure 15.

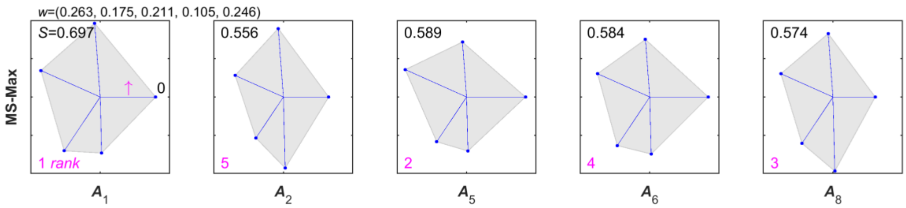

Figure 15. Deformation of a cloud of data of multidimensional objects based on a radar diagram for MS-transformation. Decision matrix D_0. Attribute weights ω=(0.263, 0.175, 0.211, 0.105, 0.246).

As an expected result, the ranking of alternatives, taking into account the weights of attributes, has changed — alternative A_5 becomes the alternative of the second rank, and alternative A_8 — the third rank.

6. Expanding the range of choice when using IZ and MS transformations for non-linear aggregation methods

This section presents the results of ranking alternatives in the WPM aggregation model for two decision matrices D_2 and D_3 (Table 5), which are highly sensitive to IZ and MS transformations, respectively.

D_2 and D_3 are generated in accordance with the methodology described in section 4.2 above, based on the D_0 matrix.

Table 5. D_2 and D_3 decision matrices with high sensitivity to IZ and MS transformations.

D_2	C_1^+	C_2^-	C_3^+	C_4^+	C_5^-	D_3	C_1^+	C_2^-	C_3^+	C_4^+	C_5^-
A_1	544,6	68,4	135,0	2226,2	5640,0		512,023	54,000	183,000	1386,692	6500,000
A_2	667,0	92,0	144,7	1056,0	6246,0		667,000	79,324	139,289	1223,729	4200,000
A_3	554,7	88,8	152,4	2288,5	4200,0		505,563	66,604	143,573	2631,228	5337,549
A_4	522,4	86,5	183,0	1514,6	4430,9		539,942	64,426	135,000	2451,162	4960,494
A_5	622,8	71,4	160,5	2680,0	6500,0		581,229	92,000	172,123	2048,350	6259,448
A_6	448,0	54,0	135,6	1287,2	5858,0		535,617	68,758	159,478	2284,541	6464,639
A_7	552,1	76,3	166,1	2040,8	6282,3		448,000	89,566	135,612	2680,000	5224,235
A_8	448,0	60,0	178,1	2518,7	5910,9		578,995	62,112	164,724	1056,000	5131,603

The ranking results using the WPM method and the values of the relative efficiency indicator of alternatives dQ in accordance with formula (4.1) are presented in the Table 6.

Table 6. Result of ranking of the alternatives for various normalization models for decision matrices D_2 and D_3. WPM aggregation method.

***D*₂**	*Norm*						IZ-*Norm*						MS-*Norm*					
Norm()	I*)	dQ_1	II	dQ_2	III	dQ_3	I	dQ_1	II	dQ_2	III	dQ_3	I	dQ_1	II	dQ_2	III	dQ_3
Max	5	1,3	8	12,7	3	9,9	5	0,8	8	9,1	3	2,9	5	5,1	8	9,3	3	5,3
Sum	5	1,3	8	12,7	3	9,9	5	0,8	8	9,0	3	2,9	5	2,6	8	11,3	3	3,8
Vec	5	1,3	8	12,7	3	9,9	5	0,8	8	9,0	3	2,9	5	3,3	8	10,8	3	4,3
Max-Min	4	2,0	3	8,2	7	89,8	4	2,0	3	8,2	7	89,8	5	8,6	3	2,9	7	1,3
dSum	8	1,1	3	8,4	4	2,5	8	0,1	5	10,5	3	2,5	5	3,8	8	10,5	3	4,6
Z-score	5	8,6	3	2,9	7	1,3	3	1,4	4	5,2	5	6,0	5	13,3	8	0,6	3	7,1
***D*₃**																		
Max	4	5,2	3	43,2	6	40,5	4	4,8	8	0,9	3	14,8	4	0,5	8	6,6	2	3,7
Sum	4	5,2	3	43,2	6	40,5	4	4,8	8	0,8	3	15,0	2	1,1	8	1,6	4	12,1
Vec	4	5,2	3	43,2	6	40,5	4	4,8	8	0,9	3	14,9	8	1,1	4	0,4	2	11,5
Max-Min	3	27,5	2	10,1	6	62,4	3	27,5	2	10,2	6	62,3	3	0,5	4	15,0	6	1,8
dSum	4	11,3	3	22,0	2	4,4	4	5,8	8	1,3	3	11,2	8	0,7	4	2,4	2	9,2
Z-score	3	0,5	4	15,0	6	1,8	3	22,7	4	0,6	8	9,4	4	5,1	3	4,9	8	11,8

*) columns I, II, III contain numbers of alternatives of 1, 2, 3 ranks, respectively. (A_i, i=1,..., m).

For four of the six options for choosing the IZ and MS transformation scale, the alternatives of the first rank are different (color highlighted in the Table 6). The high sensitivity of the rating to the values of the decision matrix (local priorities of alternatives) is characterized by a low value of dQ. This means that the alternatives are hardly distinguishable. If we take the values dQ<5% as the lower limit of distinguishability, then in 14 out of 18 variants (according to the Table 6) the alternatives of the first and second ranks for the matrix D_2 are indistinguishable. Similarly, alternatives of the first and second ranks for the matrix D_3 are indistinguishable in 9 cases out of 18. Several alternatives claim the role of alternatives of the first rank: A_5, A_3, A_4 and A_8 for the problem with the decision matrix D_2, and A_2, A_3, A_4 and A_8 for the problem with decision matrix D_3.

To expand the spectrum of analysis, we introduce the aggregation models SAW, TOPSIS(L_2), GRA, WPM, WASPAS, and COPRAS into consideration. Combining them with 6 basic linear methods of normalization allows you to perform ranking for 36 different models. Introduction to the consideration of IZ transformations in six different types of normalization measurement scales expands the range of options for non-linear aggregation methods WPM, WASPAS and COPRAS by another 18 models. Introduction to the consideration of MS transformations of normalized values in six options expands the range of options for non-linear aggregation methods WPM, WASPAS by another 12 models. The distribution diagram of the rating of alternatives for the decision matrix D_2 is present in Figure 16.

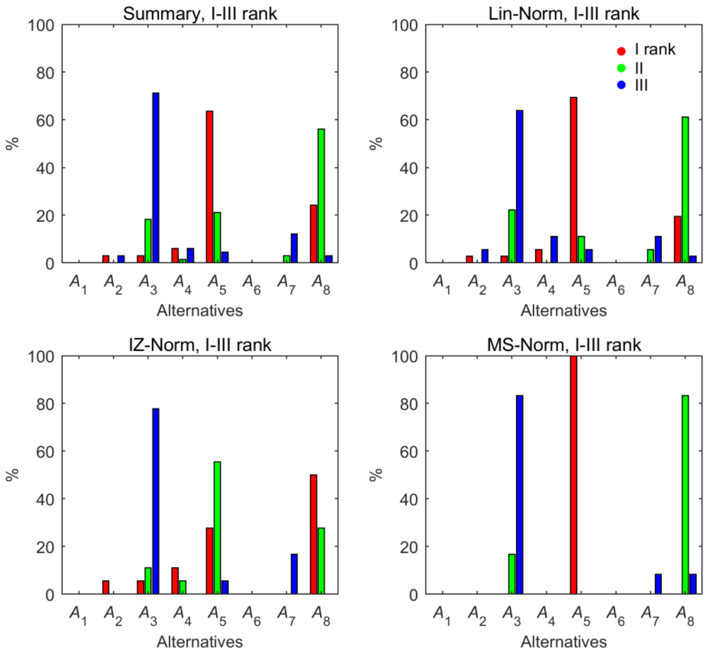

Figure 16. Distribution of rating of alternatives in 66 aggregation-normalization models. Decision matrix D_2. Equal attribute weights.

Given the absence of a formal criterion for assessing the priority of any normalization method for decision matrix, or attribute aggregation method, the choice of an alternative is not unambiguous. As shown in Section 4.2, the scatter of results is largely determined not only by the choice of aggregation method and normalization method, but also by the local priorities of alternatives for various attributes, determined by the initial values of the decision matrix.

The result of such an analysis is to recommend to the decision maker several options. In the case of the problem defined by the matrix D_2, in accordance with the results in the Figure 16 are recommended for selection, in descending order of the share of the first places, alternatives A_5, A_8, A_3 and A_4.

7. Conclusion

Weak formalization of rank models MCDM requires a comprehensive analysis using various methods of aggregation and normalization and sensitivity analysis of decisions. A critical analysis of multivariate normalization methods allows us to conclude that in the absence of criteria, the preference for certain normalizations is relative. As shown in the article, the result of the ranking of alternatives depends on the ratio between the values of the original decision matrix and is determined by the local priorities of the alternatives according to various criteria. In such a situation, it is advisable to ensure the equality of the contributions of various criteria to the indicator of the effectiveness of alternatives. The ReS, IZ and MS transformations proposed in the article transform the normalized values to a conditionally general scale $[I; Z]$, which is an important and key aspect of matching scales for measuring individual features in multivariate data normalization. The formal criterion for such a transformation is the elimination of the priority or contribution of individual features to the integral indicator of the alternative at the stage of normalization. All three methods use an independent transformation of the normalized values of various attributes. Therefore, the limitation of the application is the interdependence of the criteria. The ReS-algorithm is universal when inverting the range of cost criteria for all normalization methods. IZ and MS methods is relevant for non-linear methods of feature aggregation. When using MS-transforms for MCDM tasks, it is necessary to make sure that there are no significant outliers in the original set observations. The choice of scale factors or the range of normalized values when performing the proposed transformations is ambiguous and requires additional research. In the present study, we relied on a meaningful interpretation of the proportion of a trait on the chosen normalization method.

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